Mathematical

Reviews

Edited by

R. P. Boas, Jr. **J. L. Doob** **W. S. Massey**

S. H. Gould, ***Executive Editor***

Chandler Davis, W. Freiberger ***and*** **J. A. Zilber,** ***Editorial Consultants***

H. A. Pogorzelski, ***Editorial Associate***

Vol. 19, No. 9 Oktober, 1958 pp. 933-1030

TABLE OF CONTENTS

AUTHOR INDEX

(*Continued on cover 3*)

Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) 4(82) (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place.

Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

MATHEMATICAL REVIEWS

Published monthly, except August, by

The American Mathematical Society, 190 Hope St., Providence 6, R.I.

Sponsored by

The American Mathematical Society
The Mathematical Association of America
The Institute of Mathematical Statistics
The Edinburgh Mathematical Society
Société Mathématique de France
Dansk Matematisk Forening
Het Wiskundig Genootschap te Amsterdam
The London Mathematical Society
Polskie Towarzystwo Matematyczne
Unión Matemática Argentina
Indian Mathematical Society
Unione Matematica Italiana
The Society for Industrial and Applied Mathematics

Editorial Office

Mathematical Reviews, 190 Hope St., Providence 6, R.I.

Subscription: Price $50 per year ($25 per year to members of sponsoring societies). Checks should be made payable to Mathematical Reviews. Subscriptions should be addressed to the American Mathematical Society, 190 Hope St., Providence 6, R.I.

The preparation of the reviews appearing in this publication is made possible by support provided by a grant from the National Science Foundation and by a contract with the Air Force Office of Scientific Research, Air Research and Development Command, United States Air Force. The publication was initiated with funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers or proprietors of the publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Published for the American Mathematical Society by N.V. Erven P. Noordhoff, Uitgeversmaak, Groningen, the Netherlands.

Mathematical Reviews

Vol. 19, No. 9 OCTOBER, 1958 pp. 933-1030

FOUNDATIONS, THEORY OF SETS, LOGIC

Ehrenfeucht, Andrzej. Two theories with axioms built by means of pleonasms. J. Symb. Logic 22 (1957), 36–38.

Affirmative answers are obtained for two open questions: (1) Does there exist an essentially undecidable theory with a finite number of non-logical constants which contains a decidable, finitely axiomatizable subtheory? (2) Does there exist an undecidable theory categorical in an infinite power which has a recursive set of axioms? [For the evolution of these questions, cf. A. Tarski, A. Mostowski and R. M. Robinson, Undecidable theories, North-Holland Publ. Co., Amsterdam, 1953; MR 15, 384; L. Henkin, J. Symb. Logic 20 (1955), 92–93; Nederl. Akad. Wetensch. Proc. Ser. A. 58 (1955), 326–328; MR 16, 1080].

The requirements of (1) are satisfied by a theory T_1 with identity and an equivalence relation $R(x, y)$. Besides the usual axioms of identity, equivalence and extensionality, there are axioms C_{nm} $(n, m=1, 2, \cdots)$ defined in terms of formulas φ_n, which express the fact that there is an equivalence class (with respect to R) containing just n elements. Specifically, C_{nm} is $\prod_{i=1}^{n} \varphi_m$, $\prod_{i=1}^{n} \sim \varphi_m$, or $x=x$, according as $f(n)=m$, $g(n)=m$ or $f(n), g(n) \neq m$, where f and g recursively enumerate two disjoint, recursively inseparable sets. The pleonasms (i.e., repetitions) used in building the C_{nm}'s are a device for insuring the recursiveness of the set of axioms. Now T_1 is essentially undecidable, and a decidable, finitely axiomatizable subtheory is obtained, e.g., by omitting the C_{nm}'s.

To satisfy the requirements of (2), the author constructs a theory T_2 (categorical in the power $\aleph_0$) with identity, adding a recursive set of non-logical axioms β_{nm} $(n, m=1, 2, \cdots)$; β_{nm} is $\prod_{i=1}^{n} \sim\psi_m$ or $x=x$, according as $h(n)=m$ or not, where ψ_n expresses the existence of just n elements and $h(n)$ recursively enumerates a non-recursive set. T_2 also answers negatively Mostowski's unpublished question whether every undecidable theory has an undecidable complete extension. *G. F. Rose.*

Skolem, Th. Two remarks on set theory. Math. Scand. 5 (1957), 40–46.

The paper contains two notes. In the first of these the author demonstrates how the axiom of infinity may be rephrased so that, along with some of the other axioms of Zermelo's set theory, the new axiom asserts that a given class is a set. Heretofore, neither the axiom of infinity nor the axiom of choice had been put in this form. This is accomplished by defining a property $I(m)$ of sets and showing that the class of all sets m such that $I(m)$ holds is denumerably infinite. The axiom of infinity then is the statement that this class is a set.

The second note deals with the definition of ordered n-tuples as sets. The author considers advantages and disadvantages of a number of alternative definitions of the concept. Two disadvantages noted are that, under one definition, n-tuples over sets of the same type are not of homogeneous type, and, under another definition which avoids this pitfall, for instance, $(a, a, a)=(a, a)$. The question of a best definition remains open. *E. J. Cogan.*

Härtig, Klaus. Explizite Definitionen einiger Eigenschaften von Zeichenreihen. Z. Math. Logik Grundlagen Math. 2 (1956), 177–203.

This article is concerned with the interdefinability of certain familiar relations on strings of symbols, particularly substitution (in various forms) and concatenation. [Cf. W. V. Quine, J. Symb. Logic 2 (1937), 113–119; H. Hermes, Semiotik, Hirzel, Leipzig, 1938.] Using $Z, Z_0, \cdots$ for such strings, Sub $Z_1Z_0Z_{00}Z_2$ means "setzt man in Z_1 für Z_0 überall Z_{00} ein, so entsteht Z_2", and Vk $Z_1Z_2Z_3$ means $Z_3=Z_1Z_2$. An explicit and elementary definition of Sub is given in terms of Vk, and vice versa. An elementary definition is made up only of the Z's, the relation in terms of which the definition is said to be given (here Sub or Vk), the sentential connectives 'or', 'not', 'and', etc., the quantifiers 'there is' and 'for all', and the equality sign. Simultaneous substitution can be explicitly and elementarily defined in terms of Sub (or of Vk).

The connection of Ers — Ers $Z_1Z_0Z_{00}Z_2$ meaning "ersetzt man in Z_1 nach Belieben Z_0 durch Z_{00}, so kann man Z_2 erhalten" — with Sub and Vk is discussed. The non-existence of certain explicit elementary definitions is shown, e.g., of Vk in terms of Ers. Some of these negative results relate to the number of symbols from which the strings are constructed; they have been ignored in the preceding paragraph.

The final portion of the paper deals with explicit elementary definitions in connection with syntax, particularly that of the ordinary propositional calculus. There is given, and attributed to Schröter, a simple elementary explicit criterion that a formula of the propositional calculus be well-formed. *W. W. Boone* (Manchester).

Klaua, Dieter. Berechenbare Analysis. Z. Math. Logik Grundlagen Math. 2 (1956), 265–303.

In this paper, which is a doctoral dissertation, the author carries out the details necessary to the foundations of what he calls 'computable analysis'. The work is divided into four main sections: 1. computability in the rationals; 2. computable real numbers; 3. computable real sequences, functions, and properties; 4. computable continuity, differentiability, and integrability. A computable sequence (x_n) of rationals is one such that there are general recursive functions $\psi_1(n)$, $\psi_2(n)$, $\chi_1(n)$, $\chi_2(n)$ so that $\chi_1(n) \neq \chi_2(n)$ and $x_n=(\psi_1(n)-\psi_2(n))/(\chi_1(n)-\chi_2(n))$. The computable reals are defined both as equivalence classes of computably convergent computable sequences of rationals and as computable Dedekind cuts in rationals, and the two definitions are shown to lead to the same class of numbers. The statement is made that any real number for which we have effective approximation techniques turns out to be computable.

Throughout the paper, notions of computable analysis

are defined in a number of alternative ways and the relation of these definitions to one another and to the rest of the theory is closely investigated. *E. J. Cogan.*

Mostowski, A. On computable sequences. Fund. Math. **44** (1957), 37–51.

There are several definitions of "recursive real number" which are equivalent in that they give rise to the same class of real numbers. All these definitions are of the form "a recursive real number is one which can be characterized by recursive functions in a certain way." Some of these characterizations are stronger than others, in the sense that they convey more information about the characterized number; there is no generally effective method for obtaining the stronger characterization, given the weaker one. (Proofs of equivalence take the form of showing several methods which can be applied to the weaker characterization, such that, in each individual case, at least one of the methods yields the stronger characterization. However, one cannot in general decide which method has worked.)

In this paper, the author mentions a number of characterizations of recursive real numbers, of varying degrees of strength. He shows that there exist sequences of recursive real numbers which cannot be recursively enumerated with each number given by a stronger characterization, but which can be recursively enumerated with each number given by a weaker characterization. He then establishes the details of the partial ordering which this induces on his original set of characterizations.

These results appear to be strongly connected with such results as those of Dekker [Pacific J. Math. **3** (1953), 73–101, T2.2; MR **14**, 838] and Rice [J. Symb. Logic **21** (1956), 304–308; MR **18**, 369] concerning similar phenomena with classes of recursive and finite sets of integers. *H. G. Rice* (Pittsburgh, Pa.).

Addison, J. W.; and Kleene, S. C. A note on function quantification. Proc. Amer. Math. Soc. **8** (1957), 1002–1006.

The authors answer a query of Kleene [Bull. Amer. Math. Soc. **61** (1955), 193–213, p. 211; MR **17**, 4] by showing that it is not true that, for $k>0$, each predicate expressible in both $(k+1)$-function-quantifier forms is hyperarithmetical in predicates expressible in the k-function-quantifier forms. In order to accomplish this, the hierarchies $\mathfrak{N}_0, \mathfrak{N}_1, \mathfrak{N}_2, \cdots$ and $\mathfrak{L}_0, \mathfrak{L}_1, \mathfrak{L}_2, \cdots$ of XXIX of the paper cited above are compared to show that, for each $k>0$, $\mathfrak{L}_{k+1}$ is expressible in both $(k+1)$-function-quantifier forms, and is not hyperarithmetical in k-function-quantifier forms. Further, it is shown that the entire hierarchy $\mathfrak{L}_2, \mathfrak{L}_3, \mathfrak{L}_4, \cdots$ lies properly between $\mathfrak{N}_1$ and $\mathfrak{N}_2$ in hyperdegree, along with its extension at least up through the third constructive number class. *E. J. Cogan.*

Specker, E. Eine Verschärfung des Unvollständigkeitssatzes der Zahlentheorie. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1041–1045, LXXXVII. (Russian summary)

Let Z be a system of number theory which contains predicate logic and recursive function theory. The following theorem is proved in this paper: There exists no complete and consistent extension $Z*$ of Z such that the theorems which hold in $Z*$ form a set belonging to the field of sets $\mathfrak{K}$ which is generated from recursively countable sets. The proof is based on the construction in Z of a formula $A(a)$ such that for each recursive, single-valued monotone function r there is a numeral $\mathfrak{n}$ such that both the following statements are provable in Z:

$$(1)\quad A(r(0))\vee A(r(1))\vee\cdots\vee A(r(\mathfrak{n})),$$

$$(2)\quad \sim A(r(0))\vee\sim A(r(1))\vee\cdots\vee\sim(A(r(\mathfrak{n})).$$

The theorem then follows from a result of Markwald [Arch. Math. Logik Grundlagenforsch. **2** (1956), 78–86; MR **18**, 1], which states that for each set K of $\mathfrak{K}$ either K or its complement contains an infinite recursive set. *E. J. Cogan* (Bronxville, N.Y.).

Kreisel, G.; und Putnam, H. Eine Unableitbarkeitsbeweismethode für den intuitionistischen Aussagenkalkül. Arch. Math. Logik Grundlagenforsch. **3** (1957), 74–78.

Łukasiewicz hat vermutet [Nederl. Akad. Wetensch. Proc. Ser. A. **55** (1952), 202–212; MR **14**, 4], dass die Theoreme des intuitionistischen Aussagenkalküls (IAK) die einzige konsistente Klasse von Formeln bilden, welche: (i) abgeschlossen ist in Bezug auf Einsetzung und Abtrennung; (ii) A oder B enthält wenn sie $A\vee B$ enthält; (iii) alle Theoreme von IAK enthält. Die Verfasser widerlegen diese Vermutung indem sie zeigen, dass die Klasse K der Formeln, die durch den IAK abgeleitet werden können aus Formeln, die durch Einsetzung in der Formel

$$(*)\qquad (\neg p\rightarrow(q\vee r))\rightarrow((\neg p\rightarrow q)\vee(\neg p\rightarrow r))$$

hervorgehen, die Eigenschaften (i), (ii), (iii) besitzt. Hierdurch ist auch Satz 1 von Pil'čak [Ukrain. Mat. Ž. **4** (1952), 174–194; MR **17**, 932] widerlegt. *A. Heyting.*

Lightstone, A. H.; and Robinson, A. Syntactical transforms. Trans. Amer. Math. Soc. **86** (1957), 220-245.

The Uniform Predicate Calculus is defined as follows: for any positive integers m and i, x_i^m is a variable of the class m and ϕ_i^m is a functor of the class m (with the number of argument places indicated only in use); any variable is a term and any functor with its argument places filled by terms is a term of the same class as the functor; a generating wff is any sequence of terms such that the sequence of class numbers of which the terms are members belongs to a given set (called the generating set) of finite sequences of positive integers smaller than some given integer; any generating wff is a wff, and if $\mathscr{A}$ and $\mathscr{B}$ are wffs then so are $\sim(\mathscr{A})$, $(\mathscr{A})\vee(\mathscr{B})$, $Ax_i^m(\mathscr{A})$ and $Ex_i^m(\mathscr{B})$ for any i and m. Well-known axioms and rules of deduction for the functional calculus are used in the Uniform Predicate Calculus (the rule 2.10 in the authors' formulation must be dropped).

A wff is a member of the class $\mathscr{Z}$ if and only if it is of the form:

(1) $Ex_{i_1}^m\cdots Ex_{i_s}^m Ax_{j_1}^m\cdots Ax_{j_t}^m Ex_{k_1}^m\cdots Ex_{k_u}^m(\mathscr{A})$,

where $\mathscr{A}$ is a quantifier-free wff with no free variables of the class m other than the ones indicated as being quantified. The syntactical transform (mapping of some wffs into themselves) N is defined for wffs in prenex normal form: to obtain $N(Y)$ from Y, interchange the quantifiers of Y (A for E and E for A) and replace the matrix of Y by its negation. The syntactical transform Z is defined here for members of $\mathscr{Z}$ (but does not yield members of $\mathscr{Z}$) as follows: If Y is (1) then $Z(Y)$ is:

$$Ax_k^r Ex_j^r Ax_i^r Ex_{i_1}^m\cdots Ex_{i_s}^m Ax_{j_1}^m\cdots Ax_{j_t}^m Ex_{k_1}^m\cdots Ex_{k_u}^m(\mathscr{A}^*),$$

where r is distinct from any superscript in Y, and $\mathscr{A}^*$ is

$$(x_{i_1}^m x_i^r\wedge\cdots\wedge x_{i_s}^m x_i^r\wedge(x_{j_1}^m x_j^r\wedge\cdots\wedge x_{j_t}^m x_j^r\supset(x_{k_1}^m x_k^r\wedge\cdots\wedge x_{k_u}^m x_k^r\wedge\mathscr{A}))).$$

(Here $x_i^m x_k^r$ can be understood as asserting that x_i^m is a member of the class x_k^r.) For wff X such that $N(X) \in \mathscr{Z}$, $Z(X)$ is defined to be $N(Z(N(X)))$. Let S_z be the conjunction of $Ax_i^r Ax_j^r(x_i^r x_j^r \vee x_j^r x_i^r)$, $Ax_i^r Ax_j^r Ax_k^r(x_i^r x_j^r \wedge x_j^r x_k^r \supset x_i^r x_k^r)$, $Ax_i^m Ax_j^r Ax_k^r(x_i^m x_j^r \wedge x_j^r x_k^r \supset x_i^m x_k^r)$, and $Ax_i^r Ex_j^m(x_j^m x_i^r)$.

The main theorem for the syntactical transform Z is: (2) for any $Y, X_1, \cdots, X_n$ for which $Y, N(X_1), \cdots, N(X_n) \in \mathscr{Z}$, if $X_1 \wedge \cdots \wedge X_n \supset Y$ is provable in the Uniform Predicate Calculus, then so is $S_z \supset (Z(X_1) \wedge \cdots \wedge Z(X_n) \supset Z(Y))$. An application of this theorem yields: under a certain interpretation of the wffs, if X is any wff of the class $\mathscr{Z}$ holding in any completely divisible torsion-free abelian group, then $Z(X)$ holds in any ordered completely divisible abelian group.

Three other syntactical transforms U, T and V are defined, and the three theorems obtained from (2) by replacing Z everywhere by U, T and V are proven. Applications of these three theorems are given.

Some of the troublesome misprints and errors are: the inclusion of 2.10; none of the examples of special cases of the Uniform Predicate Calculus at the top of page 223 are valid without limitations; in 5.9, for the second subscript j_1 read j_i and for the subscript k_i read k_n; and in line 11 from the bottom of page 231 read "Z" for "C".

P. C. Gilmore (Ossining, N.Y.).

Krasner, Marc. Théorie de la définition. I. J. Math. Pures Appl. (9) **36** (1957), 325–357.

After a long introduction a formal system is presented which is intended to embody the ideas of Kronecker on the foundations of mathematics. As the presentation of the system is unconventional, an evaluation of it is difficult.

P. C. Gilmore (Ossining, N.Y.).

ALGEBRA

Hashimoto, Junji. Direct, subdirect decompositions and congruence relations. Osaka Math. J. **9** (1957), 87–112.

This paper is devoted to the study of the structure of an (universal) algebra in terms of the associated lattice of congruence relations. Two new concepts are introduced for this purpose. If $\{A_\alpha\}$, for $\alpha \in \Omega$, is a family of (universal) algebras of a given type, and if L is an ideal in the Boolean algebra of all subsets of Ω, then a subalgebra S of the complete direct union $\prod (A_\alpha)$ is an 'L-restricted direct union' of the A_α if 1) $\{x_\alpha\}, \{y_\alpha\} \in S$ imply $\{\alpha | x_\alpha \neq y_\alpha\} \in L$, and 2) $\{x_\alpha\} \in S$ and $\{\alpha | x_\alpha \neq y_\alpha\} \in L$ imply $\{y_\alpha\} \in S$. A family of congruence relations on an (universal) algebra A is 'completely permutable' if for every subcollection $\{\theta_\nu\}$, $x_\lambda \equiv x_\mu\ (\varphi_\lambda \cup \varphi_\mu)$, where $\varphi_\lambda = \bigcap \theta_\nu$ for all $\nu \neq \lambda$, implies $x \in A$ exists such that $x \equiv x_\nu\ (\theta_\nu)$ for all ν. Let $\Theta(A)$ denote the lattice of congruence relations on A. The author shows that L-restricted direct union representations of an algebra A are in one-to-one correspondence with closed sublattices of $\Theta(A)$ containing the unit and null congruence relations and such that the principal congruence relations are completely permutable and form a lattice isomorphic to L. In particular, the complete direct union representations can be characterized in terms of complete permutability. When L is the ideal of finite subsets, this theorem gives a characterization of the usual restricted direct union representations in terms of complete permutability.

Unicity of representations into indecomposable and simple factors is also discussed. Typical results are the following: If 1) A has a one-element subalgebra, 2) all congruence relations on A are permutable, and 3) all decomposition congruence relations on A form a sublattice of $\Theta(A)$, then, for any two representations of A as a subdirect union of indecomposable factors, there is a one-to-one correspondence between the two sets of factors such that corresponding factors are isomorphic in pairs. If A is an algebra for which all congruence relations permute, then A can be represented as a finitely restricted direct union of simple factors if and only if $\Theta(A)$ is complemented. Moreover, if A contains a one-element subalgebra, the factors are unique to within isomorphism.

The author also discusses algebras for which $\Theta(A)$ is distributive and direct decompositions of complete lattices. He concludes with a theorem which generalizes slightly a theorem of the reviewer on the representation of relatively complemented lattices as direct unions of simple lattices.

R. P. Dilworth (Pasadena, Calif.).

Combinatorial Analysis

Stanton, R. G.; and Sprott, D. A. A family of difference sets. Canad. J. Math. **10** (1958), 73–77.

Let p, q be primes in the relation $p^n+2=q^m$, and let the abelian group A, of order $v=p^nq^m$, be the direct sum of the additive groups of the finite fields $GF(p^n)$ and $GF(q^m)$. The authors show explicitly how to select $k=(v-1)/2$ elements of A which form an abelian difference set of type (v, k, λ), where $\lambda=(v-3)/4$. They suppose that generators x, y of the multiplicative groups of $GF(p^n)$, $GF(q^m)$, respectively, are at hand. Then the difference set consists of $(0, 0)$, the elements (x^i, y^i) and the elements $(x^j, 0)$, where $0 \leq i \leq (p^{2n}-3)/2$, $0 \leq j \leq p^n-2$.

Three examples are given. By comparing their abelian (63, 31, 15) design with the two cyclic designs with these parameters which are listed by M. Hall [Proc. Amer. Math. Soc. **7** (1956), 975–986; MR **18**, 560], the authors show that their design is new, and hence (assuming Hall's list is complete) does not have a cyclic representation.

R. H. Bruck (Madison, Wis.).

★ **Riley, John A. Occupancy theory with application to multichannel communication systems. I. Theory.** Communication Laboratory, Air Force Cambridge Research Center, Air Research and Development Command, Parke Mathematical Laboratories, Inc., Carlisle, Mass., Sci. Rep. No. 5, vii+49 pp. (1957).

This report is concerned with the distribution of objects of specification (n^N), that is, of N kinds and n of each kind, into M different cells, with no more than m like objects in any cell and without regard to ordering within cells. The distributions are classified in the first place according to the total number of objects in each cell, without regard to kind of object, that is, as to how many ways the objects may be distributed so that cell i contains a_i objects, $i=1, 2, \cdots, M$. Next, they are classified according to the number of objects of each kind in a given cell. Finally, they are considered according to the number of cells in a given group of k, $k \leq M$, which contain exactly a objects. These three are called "fundamental" problems and are stated in probability language, as is also a fourth problem, that of finding the expected number of objects per cell. The solution reached for the first, following a procedure modelled on Markov's treatment of random flights, is that the multivariable generating function of the distribution,

$G(x_1, x_2, \cdots, x_M)$, with x_i the indicator of occupancy of cell i, is given by $[\sum_{s=0}^{n} a_s(z_1, z_2, \cdots, z_M)]^N$, with $a_s(z_1, \cdots, z_M)$ the elementary symmetric function having $\binom{M}{s}$ terms, $g=n/m$ or $[n/m]+1$ (with brackets indicating integral part) according as m does or does not divide n, and $z_i=x_i+x_i^2+\cdots+x_i^m$. For $m=1$, this agrees with a result given by MacMahon [Combinatory analysis, vol. I, Cambridge, 1915, sect. I, Ch. II], namely $a_n^N(x_1, \cdots, x_M)$; but for $m\geqq n$, i.e., unrestricted distribution, it does not agree with the corresponding result of MacMahon, namely $h_n^N(x_1, \cdots, x_M)$, with h_n the "homogeneous product sum" symmetric function. For other values of m, MacMahon uses the notation t_n to indicate the truncation from h_n of all monomial symmetric functions corresponding to partitions with parts greater than m, so that $t_n^N(x_1, \cdots, x_M)$ is the generator; this also is in disagreement. The second problem is solved by a simple argument which is actually independent of the first, but the solution of the third problem is not. *J. Riordan.*

Austin, T.; Fagen, R.; Lehrer, T.; and Penney, W. The distribution of the number of locally maximal elements in a random sample. Ann. Math. Statist. **28** (1957), 786–790.

Dans une permutation P des chiffres $1, 2, \cdots, n$, considérons une séquence de k termes; soit x le plus grand élément dans cette séquence, alors x s'appelle un k-maximal-élément de P. Soit i le nombre des k-maximal-éléments distincts provenant des $n-k+1$ séquences de longueur k que l'on peut prélever sur P. On se propose de calculer le nombre $f_k(n, i)$ des permutations P dont le nombre des k-maximal-éléments est i. La valeur de la fonction f peut être obtenue par induction au moyen de l'équation aux différences $f_k(n+1, i+1)=\sum_{m=0}^{n}\sum_{r=0}^{i} f_k(m, r)f_k(n-m, i-r)\binom{n}{m}$, avec les valeurs aux limites: $f_k(n, 0)=n!$, $n<k$; $f_k(n, 0)=0$, $n\geq k$; $f_k(n, i)=0$, $i>0$, $n<k$. On y parvient aussi au moyen de la fonction génératrice des nombres $f_k(n, i)/n!$. Exemples pour $k=3$ et $n=1, 2, \cdots, 8$. Les résultats s'appliquent à une population quelconque de nombres réels avec une fonction de distribution continue. *A. Sade* (Marseille).

See also: Programming, Resource Allocation, Games: Berge.

Linear Algebra

Lages Lima, Élon. An intrinsic exposition of the theory of determinants. Gaz. Mat., Lisboa **17** (1956), no. 63–64, 1–8. (Portuguese)

This is a self-contained account of the elements of determinant theory, in the spirit of the Bourbaki treatment. The author develops the minimal results required about multilinear and alternating functions. In particular he proves that if V is a vector space of dimension n there is, up to scalar factors, only one n-linear alternating function f on V. Then if T is a linear transformation on V to itself, det T may be defined by

$$f(Tx_1, \cdots, Tx_n)=\det T\cdot f(x_1, \cdots, x_n)\ (x_1, \cdots, x_n \in V).$$

On the basis of this definition some of the standard theorems on determinants are proved; (1) $\det ST=\det S\cdot\det T$; (2) $\det T\neq 0$ if and only if T^{-1} exists; (3) $\det T=\det T^*$, where T^* is the dual of T. The paper concludes with an account of the Laplace expansion. *J. H. Williamson* (Belfast).

Pearl, M. H. On Cayley's parameterization. Canad. J. Math. **9** (1957), 553–562.

The author proposes the following generalization of a well-known formula of Cayley: Let A be a real symmetric matrix (possibly singular), and Q skew-symmetric such that $|A+Q|\neq 0$. Then the matrix (1) $P=(A+Q)^{-1}(A-Q)$ has the properties (2) $P'AP=A$, $|P|=1$, and the row vectors of $I+P$ span the same space as those of A; (viz., $I+P=2(A+Q)^{-1}A$, where I is the unit matrix). Conversely, if P satisfies the conditions (2), then there is a skew-symmetric matrix Q such that P appears in the form (1). The proof of the first part is formal and simple. The proof of the second part is based on a reduction of A to a diagonal form $U'AU$ by an orthogonal U. The author also gives an example which shows that the theorem is false over a field of characteristic two. Finally, he states as "slightly simpler, but extremely similar" the "complex case" where A is supposed to be hermitian, Q skew-hermitian, and, instead of (2), one has the relation $\bar{P}'AP=A$. *H. Schwerdtfeger* (Kingston, Ont.).

Khan, N. A. Characteristic roots of the product of certain matrices. Proc. Indian Acad. Sci. Sect. A. **46** (1957), 367–370.

The main theorem concerns bounds for characteristic roots of products of commuting matrices, one of which is assumed hermitian. Frobenius' theorem concerning characteristic roots of products of commuting matrices is not used. It would yield the result more easily. *O. Taussky-Todd* (Pasadena, Calif.).

Mirsky, L. On a generalization of Hadamard's determinantal inequality due to Szász. Arch. Math. **8** (1957), 274–275.

Let A be a positive definite Hermitian matrix of order n such that at least one element off the principal diagonal is not zero. Let P_k denote the product of all k-rowed principal minors of A ($P_0=1$). The paper contains an inductive proof for Szász's result [Monatsh. Math. Phys. **28** (1917), 253–257] that the sequence $P_k^{\binom{n-1}{k-1}^{-1}}$ $(1\leqq k\leqq n)$ is strictly decreasing. The author also shows that the sequence $P_n^{-n/(n-k)}P_k^{\binom{n-1}{k}^{-1}}$ $(0\leqq k\leqq n-1)$ is strictly increasing. *Ky Fan* (Oak Ridge, Tenn.).

Bodewig, E. Zum Matrizenkalkul. II. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 301–304.

The author draws attention to a theorem of Wedderburn [Lectures on Matrices, Amer. Math. Soc. Colloq. Publ., New York, 1934, p. 69]: If A is a matrix, s and z vectors, and $A_1=A-\gamma(As)(z'A)$, where $\gamma^{-1}=z'As\neq 0$, then the rank of A_1 is one less than the rank of A. He gives a proof of this theorem which depends on the following results: $(A+sz')^{-1}=R-(1/(1+z'Rs))(Rs)(z'R)$; and $\det(A+sz')=\det A+z'A^*s$; where $R=A^{-1}$ and A^* is the adjoint of A.

The author discusses the determination of the rank of a matrix, and points out that all the known and many other methods of deflation are contained in the theorem. *B. N. Moyls* (Vancouver, B.C.).

Goldman, A. J. A matrix minimization problem. J. Washington Acad. Sci. **47** (1957), 405–406.

Let $\mathscr{C}_n$ be the set of n-square complex matrices. For $A\in\mathscr{C}_n$ denote the Frobenius norm by $N(A)=$

$(\sum_{j,k} |A_{jk}|^2)^{1/2}$. Set $F(A)=\sum_{j=1}^{4} N(A+i^jI)/(n^{1/2}+N(A))$, where $i=(-1)^{1/2}$; and set $K_n=\frac{1}{2}\min F(A)$, where the min is taken over all matrices in $\mathscr{C}_n$ for which $N(A)\leqq n^{1/2}$. The author proves that $K_n=\frac{1}{2}(1+2^{1/2})$. This is done by showing that

$$K_n=\frac{1}{2}\min\Big[\sum_{j=1}^{4}(N^2+n+2r\cos(\theta+j\pi/2))^{1/2}/(n^{1/2}+N)\Big],$$

where the min is taken over the domain defined by $0\leqq\theta\leqq 2\pi$, $0\leqq N\leqq n^{1/2}$, $0\leqq r\leqq Nn^{1/2}$ ($r=N$ if $n=1$).

B. N. Moyls (Vancouver, B.C.).

Asplund, Edgar. Metric criteria of normality for complex matrices of order less than 5. Ark. Mat. 3 (1958), 441–447.

Let F be an n-dimensional complex Hilbert space. For the linear transformation A on F let $\|A\|=\sup_{f\in F}\|Af\|\cdot\|f\|^{-1}$ be the Hilbert norm. It follows from a result of von Neumann [Math. Nachr. 4 (1951), 258–281; MR 13, 254] that if (1) $\|(p(A))^2\|=\|p(A)\|^2$, for all polynomials $p(t)$ with complex coefficients, then A is normal. The author is concerned here with transformations satisfying (1) for linear polynomials. His main result is as follows: Let Λ be a subset of the complex plane, and let (2) $\|(A-\lambda I)^2\|=\|A-\lambda I\|^2$ for all $\lambda\in\Lambda$; then A is normal (i) if $n=1$, (ii) if $n=2$ and Λ is any point, (iii) if $n=3$ and Λ is any straight line, and (iv) if $n=4$ and Λ is the whole complex plane. For the case $n=5$ a non-normal A is exhibited which satisfies (2) for all complex λ.

The set $W(A)=\{\lambda|\lambda=(Af,f)(f,f)^{-1}, f\in F\}$ is called the field of values of A. The reviewer and Marcus [Proc. Amer. Math. Soc. 6 (1955), 981–983; MR 17, 820] showed that A is normal when $n\leqq 4$ if $W(A)$ is equal to the convex hull of the spectrum of A. The author gives a neat proof of this result.

B. N. Moyls (Vancouver, B.C.).

Sahnovič, L. A. On limit values of multiplicative integrals. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 3(75), 205–210. (Russian)

Let $H(x)$ be a matrix valued function for $a\leqq x\leqq b$ and $E(t)=\int_a^t H(v)dv$. A multiplicative integral is defined as $w(x,\lambda)=$

$$\int_a^x \exp\Big(-i\frac{dE(t)}{t-\lambda}\Big)=\lim_{\max\Delta t_j\to 0}\prod_{j=1}^{n}\exp\Big(-i\frac{E(t_j)-E(t_{j-1})}{\xi_j-\lambda}\Big),$$

where $t_j\leqq\xi_j\leqq t_{j+1}$ and λ is a complex number. It has been shown [Gantmaher, The theory of matrices, Gostehizdat, Moscow, 1953; MR 16, 438] that these multiplicative integrals are matrix solutions to the systems of differential equations

$$\frac{dw(x,\lambda)}{dx}=-iw(x,\lambda)\frac{H(x)}{x-\lambda},\quad w(a,\lambda)=I.$$

Also, it is possible to express the resolvents of a certain class of non-self-adjoint operators by means of these integrals [Livšic, Mat. Sb. N.S. 34(76) (1954), 145–199; MR 16, 48].

Under the assumption that the matrix $H(t)$ is self-adjoint for each t and of uniformly bounded norm, the author investigates the limiting values of $w(x,\lambda)$ as λ goes to a point of $[a,x]$. The main result states that for almost all $\sigma\in[a,x]$ the matrices

$$w^+(x,\sigma)=\lim_{\tau\to 0+}w(x,\sigma+i\tau),\quad w^-(x,\sigma)=\lim_{\tau\to 0-}w(x,\sigma+i\tau)$$

exist and are given in terms of certain multiplicative integrals.

A second theorem proves that if $h(\sigma)$ is an eigenvalue of $w^+(b,\sigma)w^+(b,\sigma)^*$, then $(\ln h(\sigma))/2\pi$ is an eigenvalue for $H(\sigma)$.

A. Devinatz (St. Louis, Mo.).

See also: General Theory of Numbers: Vartak. **Functions of Real Variables:** Gyires. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Vidav. **Optics, Electromagnetic Theory, Circuits:** Tsetlin.

Polynomials

★ **Jordan, Camille. Traité des substitutions et des équations algébriques.** Nouveau tirage. Librairie Scientifique et Technique A. Blanchard, Paris, 1957. xviii+667 pp. 7500 francs.

A photographic reproduction of the original work of 1870.

Obreschkoff, Nikola. Über die Wurzeln von algebraischen Gleichungen mit reellen Koeffizienten. C. R. Acad. Bulgare Sci. 9 (1956), no. 3, 1–3. (Russian summary)

The following theorem is proved in this note: Let $f(x)$ be an nth degree real polynomial whose zeros $x_1, x_2, \cdots, x_n$ are arranged in increasing order. Assume that differences x_p-x_{p-1} monotonically increase with p for $p=2, 3, \cdots, n$. Then the second derivative $f''(x)$ has exactly one zero between the successive points $x_m, x_{m+1}, \cdots, x_n$, where $m=1+\frac{1}{2}n$ or $m=\frac{1}{2}(n-1)$, for n respectively even or odd. The proof is by a study of the sign changes in the function $F(x)=f''(x)/2f(x)=\sum A_k(x-x_k)^{-1}$ near the zeros x_p, $m\leqq p\leqq n-1$.

M. Marden (Milwaukee, Wis.).

Mukohda, S.; and Sawaki, S. On the $b_p^{k,j}$ coefficient of a certain symmetric function. J. Fac. Sci. Niigata Univ. Ser. I. 1 (1954), no. 2, 6 pp.

Let $\sum x_1^p x_2^p\cdots x_k^p x_{k+1}\cdots x_{j-k(p-1)}$ be a homogeneous symmetric polynomial of degree j in variables $x_1, x_2, \cdots, x_n$, where n, p, k and $j\geqq kp$ are positive integers. Express it as a polynomial in $\sigma_1, \sigma_2, \cdots, \sigma_j$ (where σ_i is the elementary symmetric function of $x_1, x_2, \cdots, x_n$ of degree i) and write the coefficient of σ_j as $b_p^{k,j}$.

The authors improve significantly on previously known results by showing that, for primes p,

$$b_p^{k,j}\equiv\binom{j-1-k(p-1)}{k}\pmod{p}.$$

In obtaining this result, they prove the formula, valid for all p,

$$\binom{m}{k}=\sum_{i=0}^{k}(-1)^{i(p-1)}\binom{m-j+i(p-1)}{k-i}b_p^{i,j},\quad m\geqq n.$$

The reviewer remarks that this formula can be inverted to yield

$$b_p^{k,j}=(-1)^{k(p-1)}\Big\{\binom{j-1-k(p-1)}{k}+p\binom{j-1-k(p-1)}{k-1}\Big\}.$$

K. Goldberg (Washington, D.C.).

Partial Order, Lattices

Schmidt, Jürgen. Die transfiniten Operationen der Ordnungstheorie. Math. Ann. 133 (1957), 439–449.

In this paper the author shows that the lattice operations (finite and transfinite) may be taken as the basic operations of partially ordered sets. Let E be a set of elements with an operation Ω, which assigns to certain subsets of E an element of E. For this operation an associative law is defined and it is proved that the ordering operations are associative in this sense. Commutativity and idempotency are defined in such a way

that the ordering operations are commutative and idempotent. The operation Ω has the following property. Let I and J be two index sets and let f and g be two single-valued mappings of I and J into E, such that the sets $\{f(i), i \in I\}$ and $\{g(j), j \in J\}$ are the same. Then Ω assigns to these two sets the same element of E. Therefore Ω is a set operation and Ω can be considered as a mapping of a subset of 2^E into E. It is shown that the ordering operations are associative normed set operations. In fact, the following theorem is proved. If Ω is an associative, invariant and idempotent operation or, what is the same, if Ω is an associative normed set operation, then $\Omega(x, y)=y$ defines in E a partial ordering and $x=\max M \Rightarrow x=\Omega M \Rightarrow x=\sup M$ $(M \subset E)$. *Ph. Dwinger.*

Ward, L. E., Jr. Completeness in semi-lattices. Canad. J. Math. **9** (1957), 578–582.

A semi-lattice (i.e., meet-closed, partially ordered set) is called complete if each of its non-empty subsets has a greatest lower bound. This paper extends to semi-lattices some well known criteria for completeness of a lattice. The first theorem generalizes a result of Frink: a semi-lattice X is complete if and only if every set of the form $L(a)=[x \in X | x \leqq a]$ is compact in the interval topology. The second main result extends a theorem of Davis and Tarski: a semi-lattice X is compact in the interval topology if and only if every isotone function $f: X \to X$ has a fixed point. {Reviewer's remark: some of the proofs in this paper can be substantially shortened by reducing them to the lattice case, using the simple fact that a complete semi-lattice becomes a complete lattice upon adjunction of a unit element.} *R. S. Pierce.*

Hájek, Otomar. Direct decompositions of lattices. I. Czechoslovak Math. J. **7(82)** (1957), 1–15. (Russian summary)

To every neutral (central) element of a lattice L, there corresponds a unique representation of L as a subdirect (direct) product of two lattices. The following results are typical: A homomorphism of a subdirect product L of two lattices corresponding to a neutral element of L can be extended to a homomorphism of the direct product of these factors. If L is a subdirect product of the lattices M_1, M_2 or N_1, N_2 corresponding to neutral elements m and n of L, respectively, then there exist the lattices L_i $(i=1, 2, 3, 4)$ with the property that every M_j, N_j is a subdirect product of two suitable lattices L_i, and L is a subdirect product of them all. Similar theorems for rings are given. Some of the results are well-known or are consequences of known results (cf., e.g., Theorem 7). *M. Novotný* (Brno).

Rieger, Ladislav. A remark on the so-called free closure algebras. Czechoslovak Math. J. **7(82)** (1957), 16–20. (Russian. English summary)

The author constructs an infinite closure algebra with one generator. It follows that the free closure algebra with one generator is infinite, which corrects a result of G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., vol. 25, rev. ed., New York, 1948, Ch. IX, § 7, Th. 8; MR **10**, 673]. *M. Novotný* (Brno).

Yablonskiĭ, S. V. On classes of functions of the algebra of logic admitting a simple schematic realization. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 6(78), 189–196. (Russian)

This paper treats questions arising in the investigation of certain classes of Boolean functions and their realizations in contact schemes. Q is called an invariant class of functions if it contains the constants 0 and 1, and if it contains, with every one of its elements, the functions obtained by replacing variables with constants, and the functions obtained by permuting variables. The structure of invariant classes is investigated, and it is shown that the class of all invariant classes has the power of the continuum. If $P_Q(n)$ is the number of functions in the class Q having exactly n arguments, then it is shown that, if Q does not contain all functions of algebraic logic, $\lim_{n\to\infty} (P_Q(n)/2^{2^n})=0$. Following this, contact systems with $P_Q(k)$ inputs and one output which will realize functions of the class Q are considered. It is shown that the number of contacts for realizing functions of n arguments belonging to Q in such systems does not exceed $2P_Q(n)$. Let $L_Q(n)$ be the least whole number such that any function of n arguments in Q can be realized by a scheme of not more than $L_Q(n)$ contacts, and $L(n)$ be the corresponding number for the class of all Boolean functions. Then a necessary and sufficient condition that $\lim_{n\to\infty}(L_Q(n)/L(n))=0$ is that $\lim_{n\to\infty} (P_Q(n))^{2^{-n}}=1$. The paper contains a number of examples of invariant classes, and a discussion of the significance of the results as applied to them. *E. J. Cogan* (Bronxville, N.Y.).

See also: Algebra: Hashimoto. **Groups and Generalizations:** Cohn. **Optics, Electromagnetic Theory, Circuits:** Povarov; Lunts.

Fields, Rings

Fuchs, L. Wann folgt die Maximalbedingung aus der Minimalbedingung? Arch. Math. **8** (1957), 317–319.

Let A be an Artinian ring (an associative, but not necessarily commutative ring which satisfies the minimum condition on left ideals). Then A satisfies the maximum condition on left ideals, if and only if A (considered as an additive group) contains no subgroup of type p^∞. This theorem is proved without using Hopkin's result [Ann of Math. (2) **40** (1939), 712–730; MR **1**, 2] that an Artinian ring with a one-sided unit satisfies the maximum condition. In the proof it is shown that any subgroup of type p^∞ belongs to the two-sided annihilator of an Artinian ring A. Hence an Artinian ring whose two-sided annihilator is zero satisfies the maximum condition. Hopkins' result is an immediate corollary. *K. G. Wolfson* (New Brunswick, N.J.).

Mori, Yoshiro. On the integral closure of an integral domain. III. On the integral closure of a Noetherian ring of finite dimension. Bull. Kyoto Gakugei Univ. Ser. B. no. **9** (1956), 1–5.

Mori, Yoshiro. On the integral closure of an integral domain. IV, V. On the theory of Noetherian rings. Bull. Kyoto Gakugei Univ. Ser. B. no. **10** (1957), 1–5; no. **11** (1957), 1–7.

Dans un ensemble de 5 mémoires [Pour les parties I, II, voir Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27** (1953), 249–256; **28** (1954), 327–328; Bull. Kyoto Gakugei Univ. Ser. B. **7** (1955), 19–30; MR **15**, 392; **16**, 212; **18**, 6] l'auteur donne une démonstration du théorème suivant, dû à M. Nagata: la clôture intégrale d'un anneau intègre noethérien est un anneau de Krull (="endlich diskret Hauptordnung"). Certains lemmes techniques du

mémoire III étant erronés, l'auteur corrige ces erreurs dans IV et V. Dans IV on trouve une liste d'énoncés aboutissant au théorème précité. Dans V l'auteur donne les démonstrations de ceux qui ne se trouvent pas dans I, II ou III. *P. Samuel* (Clermont-Ferrand).

Jans, James P. Compact rings with open radical. Duke Math. J. 24 (1957), 573–577.

Throughout this review, R denotes a compact totally disconnected ring whose Perlis-Jacobson radical N is open. If R is the (group) direct sum of a closed semisimple subring S and N, then R is said to admit a Wedderburn decomposition, and S is called a Wedderburn factor of R. Any two Wedderburn factors are isomorphic. After noting that R/N has an identity, it is shown that the following are equivalent. (1) R admits a Wedderburn decomposition. (2) If $\bar{e}$ is the identity element of R/N, then this coset contains an element of the same additive order as $\bar{e}$. (3) N has a complementary subgroup. If, moreover, R has an identity element, then R admits a Wedderburn decomposition if and only if it is the ring direct sum of a finite number of algebras over finite fields.

If, in addition, N^2 is open, then the (transfinite) powers of N form a base of neighborhoods of 0. There is a free compact ring F such that every R for which N admits a Wedderburn decomposition, and N^2 is open, is a homomorphic image of F. The paper concludes with a discussion of the segregation properties of such free rings F (cf. Jans, Nagoya Math. J. 11 (1957), 1–7; MR 19, 249). *M. Henriksen* (Lafayette, Ind.).

See also: Foundations, Theory of Sets, Logic: Lightstone and Robinson. **Partial Order, Lattices:** Hájek. **Control Systems:** Moisil.

Algebras

Gale, David. Subalgebras of an algebra with a single generator are finitely generated. Proc. Amer. Math. Soc. 8 (1957), 929–930.

The theorem of the title is obtained as a corollary of the following: If F is a field and A a subalgebra of $F[x]$ containing elements of degree n, then A has a set of $n+1$ generators. The brief argument depends essentially on the fact that if $p \in A$, then $F[x]$ is a finitely generated module over the principal ideal ring $F[p]$. *W. G. Lister.*

Behrens, Ernst-August. Eine Charakterisierung der T-Moduln mit distributivem Untermodulverband bei halbprimärem T. Arch. Math. 8 (1957), 265–273.

The author interprets the distributivity of the lattice V of all right T-submodules of a right T-module m in ring-theoretical terms provided T is semi-primary (that is, T has the minimum condition on right ideals and an identity element 1). He writes (1) $T=T_1+\cdots+T_q+N$ (supplementary sum) with T_i primary (that is, T_i/W_i is simple, W_i the radical of T_i) for $i=1, \cdots, q$ and $N\subseteq W=$ the radical of T. [See N. Jacobson, Structure of rings, Amer. Math. Soc. Colloq. Publ., vol. 37, Providence, R.I., 1956, p. 56; MR 18, 373.] Then V is distributive if and only if every factor module mW_i^k/mW_i^{k+1} is an irreducible (T_i/W_i)-module ($i=1, \cdots, q$; $k=0, \cdots, r_i$, $W_i^{r_i-1}\supset W_i^{r_i}=0$). The author then shows that a subring occurs as a primary summand in a decomposition (1) if and only if it is a maximal primary subring of T of the form eTe with e idempotent in T. This permits a second characterization of the distributivity of V which is free of the particular decomposition (1). He then applies these results to the case $m=T$ to give three necessary and sufficient conditions that the lattice V_r of all right ideals of a semi-primary ring T be distributive. For example, V_r is distributive if and only if T is a finite direct sum of completely primary rings T_i and W_i^k/W_i^{k+1} are isomorphic (T_i/W_i)-modules ($i=1, \cdots, q$; $k=0, \cdots, r_i$). This result is related to one of R. L. Blair [Trans. Amer. Math. Soc. 75 (1953), 136–153; MR 15, 4] which asserts that when T is Jacobson semi-simple, V_r is distributive if and only if T is a subdirect sum of division rings. *M. F. Smiley.*

Groups and Generalizations

★ **Zassenhaus, Hans J. The theory of groups.** 2nd ed. Chelsea Publishing Company, New York, 1958. x+265 pp. $6.00.

The first edition [Teubner, Leipzig-Berlin, 1937] of this book appeared before the existence of Mathematical Reviews and the English translation [Chelsea, New York, 1949; MR 11, 77] was only briefly noted. Nevertheless the book is too well known for the second edition to require a full scale review. The second edition exceeds the first in length by roughly 100 pages. Except as noted below, the first English edition has been left almost untouched; the new material is almost entirely in the form of appendices and, in addition, nearly half of it consists of exercises.

Chapter I, Elements of group theory, is followed by 18 new exercises dealing with double decomposition, semigroups and other concepts not discussed in the chapter. {The reviewer is sorry to see the term "groupoid", which is widely regarded at present as synonomous with "multiplicative system", used in Exercise 15 in the sense of "Brandt groupoid".}

Chapter II, Homomorphies and groups with operators, has been enlarged as follows: To § 5, Normal chains and normal series, has been added 15 pages of material dealing with lattice-theoretical aspects of the topic. (The same point of view is extended further at the end of the volume in the 29 pages of Appendix B, Structure theory and direct products.) In § 7, the old '5. $\mathfrak{S}$-rings and hypercomplex systems' has been completely rewritten as '5. Semi-modules, semi-rings, $\mathfrak{S}$-rings, algebras'. 30 exercises are added to the chapter in Appendix A.

Chapter III, The structure and construction of composite groups, is unchanged except that the two Theorems 16, and all subsequent theorems, have had their numbers increased by one. At the end of the volume, the 13 pages of Appendix C, Free products and groups given by a set of generators and a system of defining relations, are offered as an introduction to §§ 3–9 of this chapter. Appendix D adds 31 exercises.

Chapter IV, Sylow p-groups and p-groups, is unchanged. Appendices E, F add 17 exercises. Appendix G, A theorem of Wielandt, gives a proof of the well-known theorem on the automorphism tower of a finite centreless group. This theorem solved a problem which was mentioned as open in Chapter II of the first edition.

Chapter V, Transfers into subgroups, is unchanged. Appendix H adds one exercise.

The first edition of this book has enjoyed and deserved a reputation as a classic. The second edition — paradoxically, perhaps, since it contains all of the first — does not seem to this reviewer to be, in 1958, nearly so significant a

volume. It is to be hoped that the author will soon embark on a completely rewritten third edition.

R. H. Bruck (Madison, Wis.).

Berman, S. D.; and Lyubimov, V. V. Groups allowing arbitrary permutation of the factors of their composition series. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 5(77), 181–183. (Russian)

The principal theorem is this: a group allows the arbitrary permutation mentioned in the title if and only if it is the direct product of Γ-groups. Definition: a group is a Γ-group if all its factors of composition are isomorphic to one and the same simple group.

Gaschütz, Wolfgang. Gruppen, in denen das Normalteilersein transitiv ist. J. Reine Angew. Math. 198 (1957), 87–92.

Groups G with the above property, so-called t-groups, were previously studied by E. Best and O. Taussky [Proc. Roy. Irish Acad. Sect. A. 47 (1942), 55–62; MR 4, 2] and G. Zacher [Ricerche Mat. 1 (1952), 287–294; MR 14, 722]. The latter gave a characterization of all solvable t-groups. Another characterization is given here in a manner which permits the construction of all solvable t-groups as the following abelian extensions of abelian or hamiltonian groups (the author suggests the name Dedekind groups for these two classes of groups jointly): G contains a normal abelian subgroup L of odd order, prime to the order of G/L; G/L is a Dedekind group, and the inner automorphisms of G induce powers in L. In particular the case is determined when G/L is the maximum nilpotent quotient group. The construction method deduced from these results shows when two solvable t-groups are isomorphic. A simple consequence of the results is also that a subgroup of a t-group is again a t-group. Some results concerning the non-solvable case are derived, too, using the maximum solvable normal subgroup. However, the author does not think it is possible to give a complete discussion of this case at the present state of knowledge of automorphisms of simple groups.

O. Taussky-Todd (Pasadena, Calif).

Sah, Chih-Han. On a generalization of finite nilpotent groups. Math. Z. 68 (1957), 189–204.

A finite group G is called semi-nilpotent if the normalizer of any non-normal nilpotent subgroup of G is again nilpotent. The author proves the following results on the structure of a semi-nilpotent group G: 1. If $F(G)$ denotes the maximal nilpotent normal subgroup of G, then $G/F(G)$ is cyclic; hence, in particular, G is soluble. 2. $F(G)=F_0(G)H(G)$, where $F_0(G)$ is the product of all Sylow subgroups of G which are normal in G and $H(G)$ is the hypercenter of G. 3. If G is not nilpotent, then $\{F(G)\}$ and {normalizers of non-normal Sylow subgroups of G} are the complete classes of maximal nilpotent subgroups of G. Moreover, if A and B are two distinct members of the second class, then $G=F(G)A=F(G)B$ and $H(G)=F(G)\cap A=F(G)\cap B=A\cap B$.

The author also studies in detail the structure of G having trivial center, as well as the structure of a finite (semi-nilpotent) group such that the normalizer of any non-normal primary subgroup in it is again primary.

K. Iwasawa (Cambridge, Mass.).

Ribenboim, P. Conjonction d'ordres dans les groupes abéliens ordonnés. An. Acad. Brasil. Ci. 29 (1957), 201–224.

L'auteur poursuit ici les travaux de Krull, de Lorenzen et du rapporteur sur la réalisation des groupes ordonnés.

Soit P l'ensemble des éléments $\geqq 0$ du groupe abélien réticulé G. Si $x\in P$, désignons par $\bar{x}$ le filet de x. Si à tout $x\in P$, on peut associer x', $x''\in P$ tels que $x=x'+x''$, $\bar{x'}\leqq\bar{x}$ et $\inf(\bar{x''},\bar{x})=\bar{0}$, on dira que le groupe G est totalement décomposable. Lorsqu'il en est ainsi, l'auteur montre que l'on peut considérer G comme un sous-groupe ordonné du produit direct ordonné $\prod_{\alpha\in I} G_\alpha$, les conditions suivantes étant remplies:

1) Tous les G_α sont totalement ordonnés.

2) Si $x\in P$, désignons par $\sigma(x)$ le sous-ensemble de I formé par les $\alpha\in I$ tels que la projection de x sur G_α soit $\neq 0$. Alors $\sigma(x)$ ne dépend que de $\bar{x}$, et σ définit un isomorphisme de treillis de l'ensemble des filets de G dans l'ensemble de parties de I.

3) Si $\alpha,\beta\in I$ et $\alpha\neq\beta$, on peut trouver $a, b\in P$ tels que $\alpha\in\sigma(a)$, $\beta\in\sigma(b)$ et $\inf(a,b)=0$.

L'auteur donne quelques applications de ce qui précède, en particulier l'étude des préordres de G plus fins que l'ordre donné.

P. Jaffard (Lyon).

Cohn, P. M. Groups of order automorphisms of ordered sets. Mathematika 4 (1957), 41–50.

If S is an ordered set, an order automorphism of S is a 1-1 mapping α of S upon itself such that $a<b$ implies $a\alpha<b\alpha$ $(a, b\in S)$. The set of all order automorphisms of S forms a group $A(S)$. The author proves that $A(S)$ can be ordered if and only if $A(S)$ is an abelian group.

L. J. Paige (Los Angeles, Calif.).

Schwarz, Štefan. Semigroups satisfying some weakened forms of the cancellation law. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 6 (1956), 149–158. (Slovak. Russian and English summaries)

The author considers the following conditions for a semigroup S. (A) S is a sum of disjoint semigroups in each of which the cancellation laws hold. (B) For every x, y in S the equation $x^2=xy=y^2$ implies $x=y$. Then (A) implies (B). Hewitt and Zuckerman have proved that (B) implies (A) for abelian semigroups [Trans. Amer. Math. Soc. 83 (1956), 70–97; MR 18, 465]. The author gives an alternative simpler proof of this result, and a proof of the last implication for torsion semigroups. The results are applied to abelian groups.

M. Novotný (Brno).

Boccioni, Domenico. 𝔄-modulo supplementare di un S-semigruppo commutativo. Rend. Sem. Mat. Univ. Padova 27 (1957), 48–59.

The arguments of a previous paper [same Rend. 24 (1955), 474–509, 478–493; MR 17, 577] are used to prove a theorem about extensions of a semigroup with a semi-ring of operators.

H. A. Thurston (Bristol).

Tamura, Takayuki. The theory of construction of finite semigroups. II. Osaka Math. J. 9 (1957), 1–42; errata, 242.

Tamura, Takayuki. Supplement to my paper "The theory of construction of finite semigroups II". Osaka Math. J. 9 (1957), 235-237.

Continuation from Part I [same J. 8 (1956), 243–261; MR 18, 717], in which a decomposition of a semigroup by a semilattice was introduced. The present part investigates the question of whether a given semigroup can be decomposed by a given semilattice.

In the course of the paper the author studies the structure of finite semilattices, using the semigroup of translations as a tool, and showing how to obtain all semi-

lattices of a given order in terms of those of lower order. {This seems to the reviewer to be the most interesting part of the paper.} *H. A. Thurston* (Bristol).

Preston, G. B. A note on representations of inverse semigroups. Proc. Amer. Math. Soc. 8 (1957), 1144–1147.

This is the fourth in a series of papers by the author on inverse semigroups, in which for each element a there is a unique a^{-1} such that $a^{-1}a=a$ and $a^{-1}aa^{-1}=a^{-1}$. [J. London Math. Soc. **29** (1954), 396–403, 404–411, 411–419; MR **16**, 215, 216]. The concept was first introduced by V. V. Vagner [Dokl. Akad. Nauk SSSR (N.S.) **84** (1952), 1119–1122; MR **14**, 12]. Similar concepts have been investigated by Rees [Proc. Cambridge Philos. Soc. **37** (1941), 434–435; MR 3, 199] and Clifford [Ann. of Math. (2) **42** (1941), 1037–1049; MR 3, 199].

The mapping $s \to a^{-1}sa$ is an isomorphism between certain subgroups. The set $A(S)$ of such mappings is a homomorph of S. The kernel of this homomorphism is given. *H. Campaigne* (Garrett Park, Md.).

See also: Foundations, Theory of Sets, Logic: Lightstone and Robinson. **Linear Algebra:** Pearl. **Polynomials:** Jordan. **Fields, Rings:** Fuchs. **Algebraic Topology:** Mennicke. **Structure of Matter:** Fieschi.

THEORY OF NUMBERS

General Theory of Numbers

★**Сушкевич, А. К. [Suškevič, A. K.] Теория чисел: Элементарный курс. [Theory of numbers: Elementary course.]** 2nd ed. Izdat. Har'kov. Gosudarstv. Univ. im. A. M. Gor'kogo, Kharkov, 1956. 204 pp. 5.20 rubles.

There are 6 chapters, on the subjects: divisibility, Euclidean algorithm and continued fractions, congruences, quadratic residues, primitive roots and indices, quadratic forms, achievements of Russian and Soviet mathematicians. Each chapter, except the last, contains exercises.

Carlitz, L. A note on Bernoulli numbers of higher order. Scripta Math. 22 (1956), 217–221 (1957).

Let p be a prime and let r be the number of non-zero digits in the representation of k in base p. Let $B_m^{(k)}$ be the Bernoulli number of order k defined by $\sum_{m=0}^{\infty} B_m^{(k)}x^m/m!=\left(\frac{x}{e^x-1}\right)^k$. It is shown that $p^rB_m^{(k)}$ is integral (mod p) for all m. A slightly more general result is proved and made to yield the congruence

$$\sum_{\substack{m_1+m_2+\cdots+m_k=m\\ m_i>0}} \frac{(m(p-1))!}{(m_1(p-1))!\cdots(m_k(p-1))!}\equiv 0 \pmod{p^{k-r}}.$$

L. Moser (Edmonton, Alberta).

Kelisky, Richard P. Congruences involving combinations of the Bernoulli and Fibonacci numbers. Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 1066–1069.

Outline of a proof of some congruences involving sums of products of Bernoulli and Fibonacci numbers and sums of products of Bernoulli and Lucas numbers. The apparently new congruences

$$\sum_{k=0}^{p-1} B_{2k}u_{2k}\equiv \tfrac{1}{2} \pmod p \text{ if } p\equiv\pm1 \pmod 5$$

$$\sum_{k=0}^{p-3} B_{2k}u_{2(k+1)}\equiv 1 \pmod p \text{ if } p\equiv\pm2 \pmod 5$$

are special cases of these. *L. Moser.*

Johnson, J. Robert, Jr. Congruence properties of the solutions of certain difference equations. Duke Math. J. 25 (1957), 155–170.

The author considers congruence properties of the solutions of the difference systems:

(1) $u_{n+1}=f(n)u_n+cu_{n-1}$; u_0 and u_1 prescribed;

(2) $v_{n+1}=f(n)v_n+bcv_{n-1}+b^n+c^n$; $v_0=0$, $v_1=1$;

in which $f(n)$ is an odd polynomial with integral coefficients; $b, c, b^{-1}, c^{-1}, u_0, u_1$ are integral (mod m), where m is a fixed integer; $b\equiv c$ (mod p) for every prime p dividing m. For u_n, the solution of (1), and for v_n, the solution of (2), define

$$\Delta^r u_n=\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}c^{m(r-s)}u_{n+2sm},$$

$$\Delta^r v_n=\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}c^{m(r-s)}v_{n+sm}.$$

The author shows: (a) $\Delta^{2r}u_n\equiv\Delta^{2r-1}u_n\equiv 0 \pmod{m^r}$ for m odd; for m even, the modulus is $2^{-r+1}m^r$. (b) $\Delta^{2r}v_n\equiv\Delta^{2r+1}v_n\equiv 0 \pmod{m^rm'}$ for m odd, where m' is defined by $m=\prod_{p|m}p^e$, $m'=\prod_{p|m}p^{e-1}$; if m is even, the modulus is $2^{-r+1}m^rm'$. For the difference equation $u_{n+1}=f(n)u_n+g(n)u_{n-1}$, in which $f(n)$ and $g(n)$ are polynomials with integral coefficients, $f(0)=u_1$, $g(0)=c$, the author shows that, if z_n is any solution, then for all integral k and fixed integer $m\geqq 1$, $z_{n+km}\equiv u_nz_{km}+cw_nz_{km-1}$ (mod m), where u_n and w_n are independent solutions of the difference equation. The author also establishes congruence properties of determinants related to the solutions of the three difference equations considered. *D. Moskovitz.*

Battaglia, Antonio. Un caso d'impossibilità dell'equazione indeterminata: $x^{2n}+y^{2n}=z^2$. Boll. Un. Mat. Ital. (3) 12 (1957), 689–694.

The equation in the title is discussed for the case in which n is a prime and x^2+y^2 is the square of a power of a prime. The argument that the equation is impossible in integers x, y, z, none zero, is valid only if z is a prime or the square of a prime. *D. H. Lehmer.*

Palamà, Giuseppe. Sulla risoluzione completa in numeri naturali dell'equazione indeterminata $x^2+mx+p=(p+m+1)y^2$, nei casi $m=1$, 2. Boll. Un. Mat. Ital. (3) 12 (1957), 636–647.

As a generalization of a recent note by Schinzel and Sierpinski [Colloq. Math. 4 (1956), 71–73; MR 17, 1055] the author gives the complete solution of the diophantine equation in the title. Since there is the initial solution $x=y=1$, all solutions can be found from the well-known theory of the Pell equation $t^2-(p+m+1)u^2=1$ with some attention to parity when $m=1$. The author does not explain why he restricted m to the values 1 and 2. There is, as a matter of fact, no difficulty to be encountered for a general integer m. *D. H. Lehmer.*

Schwartz, H.; and Muhly, H. T. On a class of cubic Diophantine equations. J. London Math. Soc. 32 (1957), 379–382.

The equation referred to in the title is $x^2+y^2+z^2-axyz=b$, where all the letters stand for integers and $a>0$, $b\geqq 0$. This had been studied by A. Hurwitz in the case $b=0$ [Arch. Math. Phys. (3) **11** (1907), 185–196]. Hurwitz remarked, without proof, that similar results could be obtained for the case $b>0$. The case $b>0$ was treated, in passing, by Mordell in a recent paper [J. London Math. Soc. **28** (1953), 500–510; MR **15**, 102]; this treatment contains an oversight which is briefly corrected in the paper reviewed below. The authors of this paper give a complete treatment of the case $b>0$ along the lines of Mordell's argument and they find that Hurwitz's conjectures were correct, with a few exceptions. They show that, except for two exceptions, if a solution with at most one number x, y, or z equal to zero exists, then all such solutions can be obtained from a finite set of "fundamental" solutions by a sequence of so-called elementary operations. The exceptions are $a=1$, $b=4$ and $a=2$, $b=1$; in each of these cases there are infinitely many fundamental solutions. It is also proved that, except for the case $a=1$, $b=2$ (where the set of solutions is finite), the existence of one solution with at most one number x, y, or z equal to zero implies the existence of infinitely many solutions. *H. W. Brinkmann* (Swarthmore, Pa.).

Mordell, L. J. Corrigendum: Integer solutions of the equation $x^2+y^2+z^2+2xyz=n$. J. London Math. Soc. 32 (1957), 383.

This is a correction of the oversight in the paper mentioned above; the resulting treatment of the Diophantine equation in question is still incomplete.
H. W. Brinkmann (Swarthmore, Pa.).

Xeroudakes, George; and Moessner, Alfred. Three Diophantine systems. Euclides, Madrid 17 (1957), 63–71.

The three systems of the title are: $x^3+y^3+z^3=0$, $x^2+y^2=z^2$ in integers of $K(\sqrt{-2})$; $x+y+z=u+w$, $x^2+y^2+z^2=u^2+w^2$, $x^2+y^2=z^2$ in rational integers; $x+y+z=u+w$, $x^2+y^2+z^2=u^2+w^2$, $x^3+y^3+z^3=3xyz=u^3+w^3$, $x+y=x^2+y^2=z^2$ in rational numbers. The systems are solved by elementary methods. The solutions are presented implicitly and recursively. No argument for completeness is offered. *J. D. Swift.*

Sierpinski, W. Remarque sur "A note on triangular numbers" de M. B. Stolt. Portugal. Math. 15 (1956), 123 (1957).

A letter to the editor of Portugaliae Mathematica in which the author disclaims having proved that every positive integer is the sum of at most eleven triangular numbers Δ_{3n+1}, a result attributed to him by B. Stolt [Portugal. Math. **15** (1956), 87–88; MR **18**, 873]. The history of a related result is clarified. *L. Moser.*

Lomadze, G. A. Representation of numbers as sums of generalized polygonal numbers. I. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 22 (1956), 77–102. (Russian)

Leech, John. On the representation of 1, 2, $\cdots$, n by differences. J. London Math. Soc. 31 (1956), 160–169.

Rédei and Rényi [Mat. Sb. N.S. **24(66)** (1949), 385–389; MR **11**, 13] called a set of integers $a_1, a_2, \cdots, a_k$ a difference basis with respect to the positive integer n if every integer ν satisfying $0<\nu\leqq n$ can be represented in the form a_j-a_i. For given n, let $k(n)$ be the minimum number of elements in the basis. Rédei and Rényi proved that $\lim_{n\to\infty} k^2(n)/n$ exists and obtained numerical bounds for this limit. These bounds are slightly improved in this paper. (In the last line on page 161 the term mn must be omitted.) Earlier, the reviewer [J. Elisha Mitchell Sci. Soc. **61** (1945), 55–66; MR **7**, 47] considered difference bases whose elements are all less than n. Erdös and Gál [Nederl. Akad. Wetensch., Proc. **51** (1948), 1155–1158; MR **11**, 14] called such a basis a restricted difference basis and proved results for the minimum number of elements $l(n)$ in a restricted difference basis for n. The author found out that the results of Erdös and Gál are correct, but not the proof. He corrects the proof and extends the results by proving (a) $\lim_{n\to\infty} l^2(n)$ exists; (b) this limit is the greatest lower bound of $\{l(n)+t\}^2/(n+1)$ for $t\geqq 2$; (c) $2.434\cdots\leqq\lim_{n\to\infty} l^2(n)/n\leqq 3.348\cdots$. Finally, the author determines the exact values of $k(n)$ and $l(n)$ for a number of small values of n. *A. Brauer* (Chapel Hill, N.C.).

Haselgrove, C. B.; and Leech, John. Note on restricted difference bases. J. London Math. Soc. 32 (1957), 228–231.

In this paper the results of the preceeding are improved as follows: (b) holds for $t\geqq 1.68662\cdots$, and the right-hand side of (c) is decreased to $3.3342\cdots$. *A. Brauer.*

Vartak, Manohar N. On the Hasse-Minkowski invariant of the Kronecker product of matrices. Canad. J. Math. 10 (1958), 66–72.

Let C_r denote the leading principal minor matrix of order r of a matrix C. Let A, B be square matrices of orders m, n respectively, and put $D=A\times B$ (the Kronecker product of A and B). Set $u=rn+s$, $0\leqq r<m$, $0\leqq s\leqq n$. The author proves that

$$\text{(1)}\qquad \det D_u=(\det A_r)^{n-s}(\det A_{r+1})^{s}(\det B)^{r}\det B_s$$

provided none of the determinants $\det A_r$ vanishes. If A, B are rational symmetric matrices having no zero leading principal minor determinant, the author deduces the following formula for the Hasse-Minkowski invariant of $A\times B$:

$$\text{(2)}\quad \begin{aligned} C_p(A\times B)&=(-1,-1)_p{}^{m+n-1}\{C_p(A)\}^n\{C_p(B)\}^m\\ &\quad\times(\det A,-1)_p{}^{\frac12 n(n-1)}(\det B,-1)_p{}^{\frac12 m(m-1)}\\ &\quad\times(\det A,\det B)_p{}^{mn-1}.\end{aligned}$$

Here, p is a fixed prime and $(a, b)_p$ is the Hilbert norm residue symbol. Formula (2) is used to derive a result of Jones: $C_p(aB)=C_p(B)(a,-1)_p{}^{\frac12 n(n+1)}(a,\det B)_p{}^{n-1}$, where a is a non-zero rational; and a result of MacDuffee: $C_p(\Delta_m)=(-1,-1)_p{}^{m-1}\{C_p(B)\}^m(\det B,-1)_p{}^{\frac12 m(m-1)}$, where Δ_m is the direct sum of m B's.
M. Newman (Washington, D.C.).

See also: Polynomials: Jordan.

Analytic Theory of Numbers

Siddiqi, Omar Ali. On a function of Ramanujan. Proc. Indian Acad. Sci. Sect. A. 46 (1957), 371–376.

Let $\varphi=\sum_{n=1}^{\infty}(n^{n-1}/n!)\, t^{n-1}e^{-nt}$, $t\geqq 1$. In this paper an argument is offered in support of the statement that $(-1)^k(D^k\varphi)_{t=1+0}>0, k=0,1,2,\cdots$. Unfortunately, the discussion contains an instance of the principle of incomplete

induction which invalidates the argument. The author has proved the statement for the first three or four k's, however. *M. Newman* (Washington, D.C.).

Iseki, Shô. The transformation formula for the Dedekind modular function and related functional equations. Duke Math. J. **24** (1957), 653–662.

The author derives the functional equation

$$\sum_{r=0}^{\infty}\{\lambda_p((r+\alpha)z-i\beta)+\lambda_p((r+1-\alpha)z+i\beta)\}$$
$$=(iz)^{p-1}\sum_{r=0}^{\infty}\{\lambda_p((r+\beta)z^{-1}+i\alpha)+\lambda_p((r+1-\beta)z^{-1}-i\alpha)\}$$
$$-\frac{(2\pi z)^p}{(p+1)!}\sum_{\nu=0}^{p+1}\binom{p+1}{\nu}(iz)^{-\nu}B_{p+1-\nu}(\alpha)B_\nu(\beta),$$

where $\lambda_p(x)=\sum_{m=1}^{\infty} m^{-p}e^{-2\pi mx}$, and $B_\nu(t)$ is the Bernoulli polynomial of order ν. The parameter p is required to be a positive odd integer; z is complex, with $\Re(z)>0$; α and β are real, with $0\leqq\alpha\leqq 1$, $0\leqq\beta\leqq 1$ if $p>1$. When $p=1$ at least one of the inequalities $0<\alpha<1$ or $0<\beta<1$ must hold. The functional equation is proved by interchanging the order of summation in the double series $\sum_{m,n}\exp(2\pi i\alpha n+2\pi i\beta m)m^{-p}(zm+in)^{-1}$. When $p=1$, the above formula yields the famous transformation equation for Dedekind's modular function $\eta(\tau)$. In the case of odd $p>1$, it yields a similar transformation formula for the Lambert series $g_p(x)=\sum_{n=1}^{\infty} n^{-p}x^n(1-x^n)^{-1}$. [See Apostol, same J. **17** (1950), 147–157; MR **11**, 641.] *T. M. Apostol.*

Mikolás, Miklós. Über gewisse Lambertsche Reihen. I. Verallgemeinerung der Modulfunktion $\eta(\tau)$ und ihrer Dedekindschen Transformationsformel. Math. Z. **68** (1957), 100–110.

Let $g_p(x)=\sum_{n=1}^{\infty} n^{-p}x^n/(1-x^n)$, $(p=1,\ 2,\ \cdots)$; let $Q(x,\omega)=2\sum_{p=0}^{\infty} g_{2p+1}(x)\omega^{2p}$, $(|x|<1,\ |\omega|<1)$; and let $\tilde{Q}(\tau,\omega)=Q(e^{2\pi i\tau},\omega)$. Using the residue calculus, the author derives the following elegant functional equation:

$$\tilde{Q}\left(\frac{h+z}{k},\omega\right)-\tilde{Q}\left(\frac{h'-z^{-1}}{k},z\omega\right)=\tfrac{1}{2}[Y(\omega)-Y(z\omega)]+\log z$$
$$+2\pi i\mathfrak{S}_k^{h,1}(\omega,z\omega)+\pi i[(e(\omega)-1)^{-1}+(e(z\omega)-1)^{-1}]$$
$$-\frac{k}{2\pi i}\omega^{-1}(z\omega)^{-1},$$

where

$$Y(\omega)=\frac{\Gamma'(\omega)}{\Gamma(\omega)}-\frac{\Gamma'(-\omega)}{\Gamma(-\omega)},\ e(x)=\exp(2\pi ix);$$

$$\mathfrak{S}_k^{h,1}(\omega,z\omega)=$$
$$[e(\omega)-1]^{-1}[e(z\omega)-1]^{-1}\sum_{\lambda=0}^{k-1}e\left(\frac{z\omega\lambda}{k}+\omega\frac{\lambda h}{k}-\omega\left[\frac{\lambda h}{k}\right]\right);$$

h and k are integers, $(h,k)=1$, $hh'\equiv-1 \pmod k$; $\mathfrak{I}(z)>0$; and $\omega\neq n$, $\omega\neq nz^{-1}$ $(n=0,\ \pm1,\ \cdots)$. By equating coefficients of powers of ω, this formula yields the transformation equation for Dedekind's modular function $\eta(\tau)$, as well as the corresponding transformation formula for $g_{2p+1}(x)$. (See the preceding review.)

The author also studies the function

$$xy\mathfrak{S}_c^{a,b}\left(\frac{x}{2\pi i},\frac{y}{2\pi i}\right)=\sum_{m,n=0}^{\infty}\frac{x^my^n}{m!n!}\sum_{\lambda=0}^{c-1}\bar{B}_m\left(\frac{\lambda a}{c}\right)\bar{B}_n\left(\frac{\lambda a}{c}\right),$$

where $\bar{B}_n(t)$ is the periodic Bernoulli function of order n, and obtains, as a consequence, the reciprocity laws for Dedekind sums and their various generalizations due to Apostol, Carlitz, and Rademacher. *T. M. Apostol.*

Mikolás, M. On certain sums generating the Dedekind sums and their reciprocity laws. Pacific J. Math. **7** (1957), 1167–1178.

Put

$$s_{m,n}\binom{a\ b}{c}=\sum_{\lambda=0}^{c-1}P_m\left(\frac{\lambda a}{c}\right)P_n\left(\frac{\lambda b}{c}\right),$$

where $(a,c)=(b,c)=1$, $c>0$, and $P_m(x)$ is the Bernoulli function (that is, $P_m(x)=B_m(x)$ for $0\leqq x<1$, $P_m(x+1)=P_m(x)$). Thus $s_{m,n}$ is essentially the generalized Dedekind sum defined by the reviewer [same J. **3** (1953), 513–522; MR **15**, 12] and in the case $m=1$ by Apostol [Duke Math. J. **17** (1950), 147–157; MR **11**, 641]. Also put

$$\mathfrak{S}_c^{a,b}(x,\ y)=(e^{2\pi ix}-1)^{-1}(e^{2\pi iy}-1)^{-1}\sum_{\lambda(\mathrm{mod}\ c)}e^{2\pi i\lambda(ax+by)/c},$$

$$\mathfrak{D}_c^{a,b}(w,z)=\sum_{\lambda=1}^{c-1}\zeta(w,\ \{\lambda a/c\})\zeta(z,\ \{\lambda b/c\}),$$

where $\{x\}$ denotes the fractional part of x and $\zeta(z,u)=\sum_{n=0}^{\infty}(u+n)^{-z}$.

The principal results of the paper are the following. I. If a, b, c are positive and mutually co-prime, $0\leqq R(x)<1$, $-1<R(y)\leqq 0$, then

$$\mathfrak{S}_b^{c,a}(ax+by,\ -cx)+\mathfrak{S}_c^{a,b}(cx,\ cy)+\mathfrak{S}_a^{b,c}(-cy,\ ax+by)$$
$$=(1-e^{2\pi i(ax+by)})^{-1},$$

provided that $ax+by$, cx and cy are not integers. II. If $(a,c)=(b,c)=1$; $c>2$; w, z distinct from 0 and 1; then
$$\mathfrak{D}_c^{a,b}(w,\ z)=(c^{w+z}-1)\zeta(w)\zeta(z)+$$
$$\pi^{-1}(2c\pi)^{w+z-1}\Gamma(1-w)\Gamma(1-z)\{\cos\tfrac{1}{2}\pi(w-z)\mathfrak{D}_c^{b,a}(1-w,1-z)$$
$$-\cos\tfrac{1}{2}\pi(w+z)\mathfrak{D}_c^{b,-a}(1-w,\ 1-z)\}.$$

Theorem I is proved by contour integration. The result contains, in particular, the three-term relation for Dedekind sum proved by Rademacher [ibid. **21** (1954), 391–397; MR **16**, 14]. *L. Carlitz* (Durham, N.C.).

Bochner, S. On Riemann's functional equation with multiple Gamma factors. Ann. of Math. (2) **67** (1958), 29–41.

The functional equation

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\phi(s)=\pi^{-\frac{1}{2}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\psi(1-s)$$

is satisfied by $\phi(s)=\psi(s)=\zeta(s)$, and it is known from the work of Hamburger and others that, if $\phi(s)$ and $\psi(s)$ are Dirichlet series of a certain type, there is no other solution. Here, the more general functional equation

$$(2P\pi)^{-\frac{1}{2}s}\Delta_1(s)\phi(s)=(2P\pi)^{\frac{1}{2}s}\Delta_2(-s)\psi(-s),$$

where Δ_1 and Δ_2 are products of Γ-functions, is considered. It is shown that, according to what is assumed about $\phi(s)$ and $\psi(-s)$, this has either no solutions, or at most a finite number of solutions. *E. C. Titchmarsh.*

Val'fiš, A. Z. Additive number theory. XII. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **22** (1956), 3–31. (Russian)

Val'fiš, A. Z. Convergence abscissae of certain Dirichlet series. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **22** (1956), 33–75. (Russian)

Theory of Algebraic Numbers

Ishida, Makoto. On the divisibility of Dedekind's zeta-functions. Proc. Japan Acad. **33** (1957), 293–297.

Let k be an algebraic number field of finite degree and

K a finite extension of k; let $\zeta_k(s)$ denote the Dedekind zeta-function of k and likewise for $\zeta_K(s)$. It was conjectured by Artin that $\zeta_K(s)/\zeta_k(s)$ is an integral function of the complex variable s [Math. Ann. **89** (1923), 147–156]. The truth of this conjecture was proved for the case that K is a Galois extension of k by H. Aramata [Proc. Imp. Acad. Tokyo **9** (1933), 31–34] and by R. Brauer [Amer. J. Math. **69** (1947), 243–250; MR **8**, 567].

Now let K^* be the smallest Galois extension of k which contains K and let G and H be the Galois groups of K^*/k and K^*/K respectively. It is then possible to write G as a transitive permutation group in which H fixes one letter. In the present paper the author proves the correctness of the conjecture mentioned in the case that this permutation group contains, except for the identity, no permutation fixing more than one letter. To prove this the author uses the normal subgroup M of G which, according to Frobenius, is formed by the identity and the elements of the permutation group G that fix no letter. If K_0 is the field corresponding to M, he proves, using the method of Brauer, that $(\zeta_K(s)/\zeta_k(s))^n=\zeta_{K^*}(s)/\zeta_{K_0}(s)$, $n=[K_0:k]$. Since K^* is a Galois extension of K_0, the right hand side of this is an integral function according to Aramata and Brauer; the proof then follows since $\zeta_K(s)/\zeta_k(s)$ is already known to be meromorphic. *H. W. Brinkmann.*

Cohn, Harvey. A numerical study of Dedekind's cubic class number formula. J. Res. Nat. Bur. Standards **59** (1957), 265–271.

In this paper, class numbers are listed for 21 cubic fields $K_{a,b}=R(M^{\frac{1}{3}})$, where $M=s^3\pm1=ab^2c^3$, with ab squarefree. The conductor $k=\sqrt{(-D/3)}=ab$ if $9|a^2-b^2$; $k=3ab$ otherwise. The 21 cases are all for small s and $k<666$, including all cases of $s<8$; in no case is $s>23$. The author again obtains the class numbers 1, 3, 6, 9, 12 due to Dedekind and Cassels, and also the new class numbers 18, 21, 27 in the 6 cases with $k\geqq195$. The computational procedure for the SEAC is described in detail. In 15 cases with $k\leqq195$, a second run was completed by another method, and the resulting quadratic residue and nonresidue forms are listed. Machine overflow prevented completion of the second run in the remaining cases, but the first run was limited mostly by time considerations, 14 hours being required for $k=546$.
J. L. Selfridge (Los Angeles, Calif.).

Hunter, John. The minimum discriminants of quintic fields. Proc. Glasgow Math. Assoc. **3** (1957), 57–67.

The author rigorously establishes the fields of minimum absolute quintic discriminant. Here, three types are present with $D=14641$, -4511, 1609, depending on whether there are five, three, or one real root. This confirms all exploratory electronic calculations of the reviewer [Comm. Pure. Appl. Math. **8** (1955), 377–385; MR **17**, 88], and, moreover, the ranges of coefficients for the tested equations have been skillfully brought almost to within range of the computation just mentioned (in addition to being smaller in size). Thus the enormous volume of omitted human calculations for this paper could quite feasibly be replaced by more trustworthy machine calculations. The non-computational value of this paper lies in the use of the quadratic form $\sum_1^5|X_i|^2$, where X_i are the conjugates of the arbitrary field integers in the integral coordinates x_j. Then if m_i are the successive minima, $5m_2^4\leqq m_1\cdots m_5\leqq8|D|$, and it follows that the field contains an integer ρ for which $|\sum\rho_i|\leqq2$ and $5(\sum|\rho_i|^2)^4\leqq8|D|$. *H. Cohn* (St. Louis, Mo.).

Kawada, Yukiyosi. On a duality theorem in algebraic function fields. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1957), 467–482.

K is a finite unramified extension of an algebraic function field k of one variable over the field of complex numbers. Denote by $k_p^{*(s)}$ the multiplicative group of all $s\times s$ regular matrices with coefficients in the p-adic completion of k, p a prime divisor of k. Let $u_p^{(s)}$ stand for the group of $s\times s$ invertible matrices whose coefficients are holomorphic at p. Then the s-idèles $a^{(s)}$ of k are the vectors of elements in $k_p^{*(s)}$ whose components lie in $u_p^{(s)}$ for almost all p. An s-divisor is a coset $U^{(s)}(k)a^{(s)}$ modulo the group $U^{(s)}(k)$ of idèles with all components in $u_p^{(s)}$. Furthermore, denote by $P^{(s)}(k)$ the group of s-idèles whose components are equal to a matrix with coefficients in k. An s-divisor class of k is a double coset $U^{(s)}(k)a^{(s)}\cdot P^{(s)}(k)$. As in A. Weil, J. Math. Pures Appl. (9) **17** (1938), 47–87, the set of all divisor classes $\{d^{(s)}\}$, for all s, is made into a ring, the sum and product being diagonal arrangement and tensor product, respectively. Finally, extend the natural injection of the s-idèles of k into the group of s-idèles of K to a mapping i^{**} of $\{d^{(s)}\}$ into the corresponding ring of K.

The author then singles out in $\{d^{(s)}\}$ the divisor classes $D_0^{(s)}(K/k)$, for all s, for which $i^{**}d^{(s)}$ is a sum of s-divisor classes of degree 0 [as usual, defined by the sum of the p-orders of a representative]. The set $\{D_0^{(s)}(K/k)\}=D_0(K/k)$ is again a ring. It is for this ring that the author establishes a duality theorem akin to that of A. Weil, loc. cit. For this purpose, use is made of Weil's classic result on the equivalence classes of unitary representations $\alpha\to M(\alpha)$ of the group $\{\alpha\}=J(K/k)=F(k)/F(K)'$, $F(\cdots)$, denoting the fundamental group of a field, and the divisor classes $d^{(s)}$, $\theta^\alpha=M(\alpha)\theta$, θ a matrix with elements meromorphic on the maximal abelian unramified covering surface of K. Finally, the compact Hausdorff topology of $D_0^1(K)$ as a torus is suitably extended [using symmetric products of $D_0^1(K)$] to a topology of $D_0(K/k)$. Now, a unitary representation ϕ of $D_0(K/k)$, $\phi(d^{(s)})$ of degree s, is called continuous if $\mathrm{tr}(\phi(d^{(s)})M(\alpha))$ is continuous for every $\alpha\in J(K/k)$, and all s. The author's duality theorem proved then, essentially with the basic tools developed by Weil, loc. cit., states that each continuous representation ϕ of $D_0(K/k)$ is uniquely realized by an element $\alpha\in J(K/k)$ such that $\phi(d^{(s)})=M(\alpha)$, where M is a unitary representation determined by $d^{(s)}$ according to Weil's result quoted above. *O. F. G. Schilling* (Chicago. Ill.).

See also: Special Functions: Newman; Herrmann. **Algebraic Geometry:** Roquette.

Geometry of Numbers

Barnes, E. S. The inhomogeneous minimum of a ternary quadratic form. II. Acta Math. **96** (1956), 67–97.

Let $Q(x, y, z)$ be an indefinite ternary quadratic form with real coefficients and determinant $D\neq0$. For any real x_0, y_0, z_0 let $M(Q; x_0, y_0, z_0)$ denote the lower bound of $|Q(x+x_0, y+y_0, z+z_0)|$ for integers x, y, z. It was proved by Davenport [Acta Math. **80** (1948), 65–95; MR **10**, 101] that $M(Q; x_0, y_0, z_0)\leqq(27|D|/100)^{1/3}$, with equality for a particular form Q_1 when $x_0=y_0=z_0=\frac{1}{2}$, and it was further proved that the inequality is "isolated". This result was much improved by Barnes [Acta Math. **92** (1954), 13–33; MR **16**, 802] and is further improved in the present paper. It is now shown that $M(Q; x_0, y_0, z_0)\leqq$

$(|D|/4)^{1/3}$ except when Q is equivalent to a multiple of either Q_1 or Q_2, where Q_2 is a particular form for which $M(Q_2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})=(4|D|/15)^{1/3}$. Even for Q_1 or Q_2 the inequality remains valid except when $2x_0$, $2y_0$, $2z_0$ are all odd integers. The constant $\frac{1}{4}$ has a certain relevance in the problem, since it is known that $M(Q; x_0, y_0, z_0)\leq(|D|/4)^{\frac{1}{2}}$ if Q represents 0 for integers x, y, z not all 0. The work of the present paper is difficult and elaborate; it is based on a new inequality for a positive value of $Q(x, y, z)$ when $D<0$ [Proc. London Math. Soc. (3) **5** (1955), 185–196; MR **16**, 1002], and on the technique developed by Barnes and Swinnerton-Dyer [Acta Math. **92** (1954), 199–234; MR **16**, 802] for investigating the values of inhomogeneous indefinite binary quadratic forms. *H. Davenport.*

See also: Theory of Algebraic Numbers: Hunter.

ANALYSIS

Functions of Real Variables

★**Sikorski, Roman. Funkcje rzeczywiste. [Real functions.] Vol. I.** Monografie Matematyczne. Tom XXXV. Państwowe Wydawnictwo Naukowe, Warszawa, 1958. 534 pp. zł. 55.00.

This book contains the first two of the three parts of a treatise on the theory of functions of a real variable. Of its eleven chapters, the first two are an introductory exposition of sets, functions, classes of sets, and metric spaces; the next three chapters deal with continuity and convergence, and the remaining six are devoted to measure, integration, and differentiation. The second volume will contain chapters on function spaces, functional analysis, orthogonal and Fourier series, Fourier integral, and Schwartz' theory of distributions.

As for the general value of the book — unlike certain venerable (though outdated) Englishmen, the author has no desire to say everything, succeeding in it better, and unlike certain nameless Frenchmen, he confers mathematical maturity instead of demanding it. In spite of the modest claims in the preface, the completed work will represent an outstanding treatise and text.

By an explicitly stated preference, the author limits himself to subject matter with considerable applications. For instance, the integrals of Denjoy, Perron, Burkill, and (seemingly uncountably many) others are not introduced. On the other hand, the Lebesgue integral is treated exhaustively: definition and elementary properties, integral as a set-function, examples in Euclidean space, finite and infinite product measures, and multiple and iterated integrals, interchange with various limit operations, integration by parts and by substitution, connections with differentiability and differentiation, integration and differentiation with respect to a parameter, and so on. The RS and LS integrals are likewise treated.

The author likes to motivate his definitions and proofs, and there are plenty of examples and counter-examples, both in the text and in the several hundred valuable exercises.

Partial list of the contents of the first five chapters: sets and functions, algebra of sets, real series and sequences, Cartesian products, theory of power, Boolean rings and algebras, real-valued functions, measurability with respect to an additive structure, metric spaces, metric topology, Borel fields, F_σ and G_δ classifications, Euclidean spaces, pseudometrics, isometries and homeomorphisms, topological spaces, limits, convergence and continuity in metric spaces, oscillation, semicontinuity, uniform continuity and equicontinuity, function spaces and Weierstrass' theorem, Baire classes, category and functions, analytic representation of functions, ordinal numbers and Borel classification, transfinite induction, upper and lower Young classes.

Some more obvious omissions: Haar measure, Stone-Weierstrass theorem, Tychonov theorem, Banach fixed-point theorem. Also, there are many small misprints, and the index, though good, could be better. *Z. A. Melzak.*

Baiada, E.; e Cardamone, G. La variazione totale e la lunghezza di una curva. Ann. Scuola Norm. Sup. Pisa (3) **11** (1957), 29–71.

The authors show: (i) that a function f has finite total variation $V(f; a, b)$ on a linear interval (a, b) if and only if, (with $f(x)=f(b)$ for $x>b$, $f(x)=f(a)$ for $x<a$),

$$\int_a^b \left|\frac{f(x+h)-f(x)}{h}\right| dx$$

has a finite limit I when $h\to 0$, and that then $V(f; a, b)=I$; (ii) that another necessary and sufficient condition is that $\int_a^b [1+\{(f(x+h)-f(x))/h\}^2]^{\frac{1}{2}}dx$ have a finite limit L when $h\to 0$, and that then L is the length of the graph. The proofs are rather long and involved. It is conjectured, but not proved, that when f is not of bounded variation both limits are infinite. *L. M. Graves* (Chicago, Ill.).

Matorin, A. P. On inequalities between the maxima of the absolute values of a function and its derivatives on a half-line. Amer. Math. Soc. Transl. (2) **8** (1958), 13–17.

Translated from Ukrain. Mat. Ž. **7** (1955), 262–266 [MR **17**, 829].

Gyires, Béla. Über Determinanten, deren Elemente Integrale von Funktionen sind. Acta Univ. Debrecen **3** (1956), no. 2, 41–48 (1957). (Hungarian summary)

The author obtains various formulas for evaluating the determinants of square matrices whose elements are integrals of complex-valued functions over a fixed interval. *P. Civin* (Eugene, Oreg.).

Zaubek, Othmar. Über ein Stetigkeitskriterium für Funktionen mehrerer Veränderlichen und Verallgemeinerungen. Math. Nachr. **15** (1956), 265–292.

Let f be a real-valued function defined on a domain A in the Euclidean plane and let h be an accumulation point of A; then f is upper semicontinuous at h if it is upper semicontinuous at h on every continuously differentiable subset B of A containing h. A variety of results of this nature in which certain properties of f on A are characterized by its properties on suitably restricted subsets are studied extensively in this paper, together with generalizations thereof. The domain space in many cases need not be Euclidean. Detailed descriptions of the various results are too technical to be reported on in a review of this type. *M. R. Hestenes* (Los Angeles, Calif.).

Volpato, Mario. Sulla assoluta continuità e sulla validità della classica formula di derivazione delle funzioni composte. Rend. Sem. Mat. Univ. Padova **27** (1957), 37–47.

Nell'intervallo $R=I_1\times I_2\times\cdots\times I_n$, $I_r=(a_r\leq x_r\leq b_r)$,

dello spazio reale euclideo ad n dimensioni, è presa in esame una funzione reale $f(x_1, x_2, \cdots, x_n)$, assolutamente continua, separatamente, rispetto alle singole variabili. Si suppone che ogni derivata parziale f_{x_r}' $(r=1, 2, \cdots, n)$ sia continua rispetto alla $(n-1)$-upla $(x_1, x_2, \cdots, x_{r-1}, x_{r+1}, \cdots, x_n)$ per quasi tutti i valori x_r di I_r e soddisfi la limitazione

$$|f_{x_r}'(x_1, x_2, \cdots, x_n)| \leqq L_r(x_r),$$

ove $L_r(x_r)$ è una funzione non negativa e sommabile in I_r $(r=1, 2, \cdots, n)$. Sono poi indicate con $x_1(t), x_2(t), \cdots, x_n(t)$ n funzioni reali, assolutamente continue in uno stesso intervallo $I_t \equiv (c \leqq t \leqq d)$, ivi soddisfacenti alle limitazioni $a_r \leqq x_r(t) \leqq b_r$ $(r=1, 2, \cdots, n)$, tali inoltre che gli n prodotti $L_r[x_r(t)]x_r'(t)$ siano sommabili in I_t.

Sotto queste ipotesi, l'Autore dimostra che la funzione composta $f[x_1(t), x_2(t), \cdots, x_n(t)]$ è assolutamente continua in I_t, che vale, in quasi tutto I_t, la solita formula di derivazione

$$\frac{d}{dt} f[x_1(t), x_2(t), \cdots, x_n(t)] = \sum_{r=1}^{n} f_{x_r}'[x_1(t), x_2(t), \cdots, x_n(t)]x_r'(t),$$

che gli n prodotti $f_{x_r}'[x_1(t), x_2(t), \cdots, x_n(t)]x_r'(t)$ sono sommabili in I_t, e che risulta, in tutto I_t,

$$f[x_1(t), x_2(t), \cdots, x_n(t)] - f[x_1(c), x_2(c), \cdots, x_n(c)] = \sum_{r}^{1,n} \int_c^t f_{x_r}'[x_1(\xi), x_2(\xi), \cdots, x_n(\xi)]x_r'(\xi)d\xi.$$

Le dimostrazioni si valgono essenzialmente di precedenti risultati dell'Autore [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 30–37, 161–167, 299–306; MR **18**, 198] e di G. Scorza Dragoni [ibid. **12** (1952), 55–61; **20** (1956), 711–714; MR **13**, 831; **18**, 880].

T. Viola (Roma).

Sunyer i Balaguer, Ferran. Sur la détermination d'une fonction par ses nombres dérivés. C. R. Acad. Sci. Paris **245** (1957), 1690–1692.

According to a theorem of L. Scheeffer's [Acta Math. **5** (1884), 183–194, 279–296], a continuous real-valued function on the real line R is determined up to an additive constant whenever there is a countable subset of R on whose complement the function's upper right derivate is finite and known. The author announces that, here, "countable" may be replaced by "totally imperfect", and that this is the best possible such modification. He also gives a similar modification of W. H. Young's generalization [Proc. London Math. Soc. (2) **8** (1910), 99–116] of Scheeffer's theorem in terms of one-sided semi-continuity.

T. A. Botts (Charlottesville, Va.).

Četković, Simon. Un théorème de la théorie des fonctions. C. R. Acad. Sci. Paris **245** (1957), 1692–1694.

The author proves in detail the following theorem: If a given real-valued function on a real interval is continuous at each point of an everywhere-dense set, and also discontinuous at each point of an everywhere-dense set, then there is an everywhere-dense set at each point of which the function is continuous but non-differentiable.

T. A. Botts (Charlottesville, Va.).

Ravetz, J. R. Derivate planes of continuous functions of two real variables. Fund. Math. **44** (1957), 103–114.

The present work is based on a previous paper [Quart. J. Math. Oxford Ser. (2) **6** (1955), 9–26; MR **16**, 911], hereafter called D.A. Notation: z: generic point of a plane region R. z_0: fixed point of R. $f(z)$: continuous function. $a(z)$, $b(z)$: two continuously varying directions.

The results are the following. Th. 1: If $a(z) < b(z) < a(z) + \pi$, and $f(z)$ has finite directed derivates such that, on a set E residual on R, $D^a f = D_a f$, $D^b f = D_b f$, then at all points of a set F residual on R there exists a derivate plane to $f(z)$. Th. 2: If $a(z_0)$ and $b(z_0)$ are not diametrically opposed, the derivates of $f(z)$ are bounded in some neighbourhood of z_0, and $D^a f$ and $D^b f$ are continuous at $z=z_0$, then there exists a derivate plane to $f(z)$ at z_0.

The following is an indication of the proofs. For any pair of directions λ, μ not diametrically opposed, $L^{\lambda,\mu}$ denotes the linear function such that $L^{\lambda,\mu}(z_0) = f(z_0)$; $D^\lambda L^{\lambda,\mu}(z_0) = D^\lambda f(z_0)$; $D^\mu L^{\lambda,\mu}(z_0) = D^\mu f(z_0)$; $P^{\lambda,\mu:\theta} f(z_0)$ is the derivate of $L^{\lambda,\mu}$ in the direction θ. The upper $\lambda,\mu;\theta$ error derivate $E^{\lambda,\mu:\theta} f(z_0)$ is defined as $D^\theta f(z_0) - P^{\lambda,\mu:\theta} f(z_0)$; correspondingly for $E_{\lambda,\mu;\theta} f(z_0)$. E-version for the existence of a derivate plane to $f(z)$ at z_0: $E^{\lambda,\mu:\theta} f(z_0) \leqq 0 \leqq E_{\lambda,\mu;\theta} f(z_0)$ for all θ. The proof of Th. 1 is first reduced to the Lipschitzian case and the corresponding special case of Th. 1 in D.A. is used. The assertion in the E-version is assumed to be false. By means of standard uniformization procedures applied to $E^{a,b:\theta} f(z)$, related to evaluations from elementary plane geometry, a contradiction to Th. 4 in D.A. is obtained. To prove Th. 2 the assertion in the E-version is assumed to be false and a contradiction to the continuity at $z=z_0$ of $D^a f(z)$ and $D^b f(z)$ deduced. {Remark by the reviewer — Conditions in terms of linear derivates securing the existence of the derivate plane of $f(z)$ at $z=z_0$ can be found in Haupt, Aumann and Pauc, Differential- und Integralrechnung... [Bd. II, 2nd ed., de Gruyter, Berlin, 1950, §§ 3.3; MR **12**, 681.}

Chr. Pauc (Nantes).

See also: Foundations, Theory of Sets, Logic: Klaua. **Analytic Theory of Numbers:** Siddiqi. **Measure, Integration:** Timan; Beesack; Gagliardo. **Functions of Complex Variables:** Goldberg. **Sequences, Series, Summability:** Hsiang. **Approximations, Orthogonal Functions:** Stein.

Measure, Integration

Rohlin, V. A. Metric classification of measurable functions. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 169–174. (Russian)

Tsujimoto, Hitoshi; and Tanaka, Katsuro. On dominated sets of measures. Math. Japon. **3** (1955), 173–183.

Let X be a set and S a σ-algebra of subsets of X. Let M be a set of measures on S with the total measure 1. M is said to be dominated by another finite measure μ if $\mu(E)=0$ implies $m(E)=0$ for all $m \in M$, and uniformly dominated by μ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $m(E) < \varepsilon$ for all $m \in M$. The authors prove several theorems concerning necessary or sufficient conditions that M is (uniformly) dominated by some μ.

Y. Kawada (Tokyo).

Arnese, Giuseppe. Alcune osservazioni sull'integrale secondo Riemann-Stieltjes. Boll. Un. Mat. Ital. (3) **12** (1957), 648–651.

The Riemann Stieltjes integral used [see Picone and Viola, Lezioni sulla teoria moderna dell'integrazione,

Einaudi, 1952 pp. 67 ff; MR 14, 256] is defined on a bounded set I of Euclidean space S_r of r dimensions, for a point function $f(P)$ relative to an additive interval function $\alpha(t)$ of bounded variation on S_r, as $\lim_\sigma \sum_\sigma f(P_k)\alpha(T_k)$, $P_k \in T_k \cdot I$, where T_k are the intervals of the subdivision of S_r intersecting I, and the limit is via the directed set of successive subdivisions on the coordinate axes. The principal result of the note is that if the upper and lower integral, being the greatest and least of the limits of the approximating sums, respectively, are both finite, then there exists a subdivision σ_0 such that $\alpha(T)$ vanishes for all T of any finer subdivision σ on which $f(P)$ is unbounded. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Timan, M. F. On the evaluation of a certain integral. Ukrain. Mat. Ž. 9 (1957), 452–454. (Russian)

The following asymptotic equality is proved:

$$\int_0^\pi |K_n(t)|dt = \frac{2x}{\pi} \sum_{k=1}^n \frac{a_{n-k+1}}{kx+1} + O\{\max |a_k|\},$$

where $K_n(t) = \frac{1}{2}a_0 + \sum_{k=1}^n a_k \cos kt$ and the sequence a_k is either convex or concave.

Beesack, Paul R. A note on an integral inequality. Proc. Amer. Math. Soc. 8 (1957), 875–879.

The author gives a slight generalization of a result of K. Tatarkiewicz [Ann. Univ. Mariae Curie-Skłodowska, Sect. A. 7 (1953), 83–87; MR 16, 1004]. Let A_1 and A_2 be sets on which $F(x) \leq G(x)$ and $F(x) > G(x)$, respectively, and let $A = A_1 \cup A_2$ be a measurable set; let $F(x)$, $G(x)$ and $M(x)$ be integrable functions over A, and suppose that $\int_A G(x)dx \leq \int_A F(x)dx$ and that either $0 \leq M(x_1) \leq M(x_2)$ or $M(x_1) \leq 0 \leq M(x_2)$ hold for all $x_1 \in A_1$, $x_2 \in A_2$; then $\int_A G(x)M(x)dx \leq \int_A F(x)M(x)dx$. The author applies his result to the comparison of the first eigenvalues of two differential equations of vibrating strings. *J. Aczél.*

Gagliardo, Emilio. Un criterio di compattezza rispetto alla convergenza in media. Ricerche Mat. 6 (1957), 34–48.

Viene anzitutto osservato come, da richerche classiche di M. Fréchet [Fund. Math. 9 (1927), 25–32] e di M. Riesz [Acta Litt. Sci. Szeged 6 (1933), 136–142] si possano facilmente dedurre delle condizioni necessarie e sufficienti affinchè una successione $\{u_n(x_1, x_2, \cdots, x_m)\}$ $(n=1, 2, \cdots)$ di funzioni di m variabili definite, per fissare le idee, nel cubo $\Delta \equiv \{0 \leq x_i \leq 1 \ (i=1, 2, \cdots, m)\}$ ed ivi sommabili con esponente $p \geq 1$, sia compatta rispetto alla convergenza in media d'ordine p. Ma la determinazione di condizioni soltanto sufficienti, agevolmente verificabili, riveste particolare interesse. L'Autore, valendosi delle ricerche classiche sopra ricordate, dimostra in due modi diversi che: se esistono due costanti positive A, B tali che si abbia, per ogni n,

$$(1) \quad \int_\Delta |u_n(x_1, x_2, \cdots, x_m)|^p dx_1 dx_2 \cdots dx_m \leq A,$$

$$(2) \quad \int_{\Delta\times\Delta} \frac{|u_n(x_1, x_2, \cdots, x_m) - u_n(y_1, y_2, \cdots, y_m)|^p}{\sum_{i=1}^m |x_i - y_i|^m} dx_1 dx_2 \cdots dx_m dy_1 dy_2 \cdots dy_m \leq B,$$

dalla successione data se ne può estrarre una convergente (in Δ) in media di ordine p (verso una funzione di potenza p-esima sommabile, soddisfacente ancora alle (1), (2)). La grande generalità di questo criterio è provata facendo vedere che, quali corollari, se ne possono dedurre altri ben noti, dovuti rispettivamente a F. Rellich [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1930, 30–35], C. B. Morrey [Duke Math. J. 6 (1940), 187–215; MR 1, 209] e ad F. Cafiero [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8 (1950), 305–311; MR 12, 247]. Infine il nuovo criterio viene interpretato da un punto di vista funzionale e vengono date alcune formulazioni più generali, attenuando le ipotesi sulle funzioni u_n e sull'insieme ove esse sono definite. *T. Viola* (Roma).

Černý, Ilja. Das Hellingersche Integral. Časopis Pěst. Mat. 82 (1957), 24–43. (Czech. Russian and German summaries)

Es seien f, g zwei im Intervall $a \leq x \leq b$ erklärte Funktionen, für welche aus $g(x_1) = g(x_2)$ immer $f(x_1) = f(x_2)$ folgt, $p > 1$ eine Zahl. Die Intervalfunktion $\dfrac{|f(\alpha)-f(\beta)|^p}{|g(\alpha)-g(\beta)|^{p-1}}$ setzen wir gleich Null, wenn $g(\alpha) = g(\beta)$ ist. Zu dieser Intervallfunktion wird auf übliche Weise das Integral $\int_a^b \dfrac{|df(x)|^p}{|dg(x)|^{p-1}}$ und die Variation $\operatorname{var}_a^b (f, g)$ definiert [vergl., z.B., F. Riesz und B. Sz.-Nagy, Leçons d'analyse fonctionnelle, Akad. Kiadó, Budapest, 1952, S. 20, 87; MR 14, 286]; ersteres wird das Hellingersche Integral genannt. Ist g eine monotone Funktion und existiert das Integral $\int_a^b \dfrac{|df(x)|^p}{|dg(x)|^{p-1}}$, so ist es der Variation gleich; im allgemeinen sind sie voneinander verschieden. Es werden einige zur Existenz des Integrals genügende Bedingungen und einige Eigenschaften der Funktionen f, g angegeben, die sich aus der Existenz des Integrals ergeben. Für den Fall einer nicht abnehmenden Funktion g werden die Bedingungen abgeleitet, unter denen $\int_a^b \dfrac{|df(x)|^p}{|dg(x)|^{p-1}}$ auf das Lebesguesche oder Riemannsche Integral zurückgeführt werden kann. Unter den Literaturangaben fehlt der Hinweis auf eine Arbeit von J. Radon [Akad. Wiss. Wien, S.-B. IIa. 122 (1913), 1295–1438], wo ähnliche Probleme untersucht wurden. *M. Novotný* (Brno).

Krickeberg, K. Convergence of martingales with a directed index set. Trans. Amer. Math. Soc. 83 (1956), 313–337.

Let μ be a strictly positive finite measure on a Boolean sigma-algebra B. Let Θ be a non-empty index set, directed by $\ll$. Let (B_σ), $\sigma \in \Theta$, be an increasing sequence of Boolean sigma subalgebras of B, i.e., $\sigma \ll \tau$ implies $B_\sigma \leq B_\tau$, and let ϕ_σ be a real-valued countably additive function on B_σ. The sequence (ϕ_σ) is called a martingale relative to B_σ if $\sigma \ll \tau$ and $A \in B_\sigma$ imply $\phi_\sigma(A) = \phi_\tau(A)$. The associated sequence (f_σ) of Radon-Nikodym derivatives of ϕ_σ with respect to μ on B_σ is also called a martingale. If (B_σ) satisfies a certain condition V_0, and (f_σ) is a martingale, the following five conditions are equivalent: I. (f_σ) is terminally uniformly integrable. II. (f_σ) converges to a summable function f_∞, and $\lim \int |f_\sigma - f_\infty| d\mu = 0$. III. (f_σ) converges to a summable function f_∞, and $\lim \int |f_\sigma| d\mu = \int |f_\infty| d\mu$. IV. f_σ converges to a summable function f_∞, and $\lim \phi_\sigma(A) = \phi_\infty(A)$ for every $A \in \bigcup_\sigma B_\sigma$. V. (f_σ) converges to a summable function f_∞, and (f_σ) is a martingale for $\sigma \in \Theta_\infty$, where Θ_∞ is Θ with a terminal element ∞ added.

Analogous results are obtained for martingales terminally uniformly integrable from above, for μ sigma-finite on every B_σ, and for semi-martingales.

The condition V_0 is that for every $A \in B$ and (K_σ) with $K_\sigma \in B_\sigma$ and $\bigcup_{\sigma\in\Delta} K_\sigma \geq A$ for every terminal Δ, and

every $\varepsilon>0$, there are a finite set $\xi_1, \cdots, \xi_r$ of elements of Θ and disjoint sets $L_1, \cdots, L_r$ such that $L_i \leqq K_{\xi_i}$, $L_i \in B_{\xi_i}$, and $\mu(A-\cup L_i)<\varepsilon$. The condition V_0 is independent of μ, and is satisfied if (B_σ) is totally ordered by $\leqq$.

D. Blackwell (Berkeley, Calif.).

Krickeberg, Klaus. Stochastische Konvergenz von Semimartingalen. Math. Z. **66** (1957), 470–486.

Let Θ be a partially ordered set such that to every pair of elements corresponds a later element. Let μ be a strictly positive sigma-finite measure defined on a Boolean sigma-algebra. If to each σ in Θ there corresponds an extended real-valued measurable function f_σ, then the family (f_σ) is a generalized stochastic process, and the definitions of martingales and semimartingales are made in the natural way. The family (f_σ) is said to be of bounded variation if $\int |f_\sigma| d\mu$ defines a bounded function of σ on some terminal set. In the paper reviewed above, generalizing the known convergence theorems for Θ a linear set with the usual ordering, the author proved that, under a certain additional "Vitali" covering hypothesis, V_0, a martingale of bounded variation is convergent (order convergent). In the present paper he proves that, even without any such hypothesis, a semimartingale of bounded variation converges stochastically (in measure). Moreover, under a slight strengthening of V_0, it is proved that such a semimartingale even converges in the order sense. Stochastic convergence is defined by means of natural definitions of stochastic superior and inferior limits. The appropriate generalization of the Fatou theorem on integration to a limit is proved. Corresponding to the fact that the ordinary limit superior may be essentially larger than the stochastic limit superior, the hypothesis of this form of the Fatou theorem is slightly stronger than that for order convergence. This distinction disappears if Θ is the set of natural numbers in the usual order.

J. L. Doob (Urbana, Ill.).

See also: Differential Geometry: Lopšic.

Functions of Complex Variables

Roux, Delfina. Sopra-emisimmetria di tratti con eccezioni e teorema di Fabry. Boll. Un. Mat. Ital. (3) **12** (1957), 627–635.

Suppose that the power series $\sum a_n z^n$ has radius of convergence 1. Let there exist an increasing sequence of indices $n_h \to +\infty$, a sequence of complex numbers ω_h of unit modulus, and a real number θ with $0<\theta<1$, such that, setting $I_h=\{m:(1-\theta)n_h<m<(1+\theta)n_h\}$ for $h=1, 2, 3, \cdots$, (A) $|\Re(a_{n_h}\omega_h)|^{1/n_h}\to 1$; (B) as the integer u ranges over the interval $0\leqq u\leqq[\theta n_h]$, the number of variations in sign of $\Re(a_{n_h+u}\omega_h)$ is $o(n_h)$; (C) $|\Re(a_{n_h-u}\omega_h)|\leqq|\Re(a_{n_h+u}\omega_h)|$ for $u\geqq 0$, $n_h\mp u \in I_h$, $h=1, 2, 3, \cdots$. Then the point 1 is a singular point of the power series. The same conclusion holds if (B) and (C) are replaced by: (B*) v_h, the number of variations in sign of $\Re((a_{n_h-u}+a_{n_h+u})\omega_h)$ $(u=0, 1, 2, \cdots, [\theta n_h])$ is $o(n_h)$, and (C*) there exists an increasing function $\psi(n)\to+\infty$ such that, for $n_h\mp u\in I_h$, $h=1, 2, 3, \cdots$, the two real numbers

$$\Re((a_{n_h-u}+a_{n_h+u})\omega_h),$$

$$\Re((a_{n_h-u}+a_{n_h+u}\exp\{u(v_h+\psi(n_h))/n_h\})\omega_h)$$

have the same sign. *F. Bagemihl* (Notre Dame, Ind.).

Cowling, V. F.; and Thron, W. J. Zero-free regions of polynomials. J. Indian Math. Soc. (N.S.) **20** (1956), 307–310 (1957).

The authors continue a previous investigation [Amer. Math. Monthly **61** (1954), 682–687; MR **16**, 693] concerned with the polynomial $P(z)=a_0+a_1z^{\lambda_1}+\cdots+a_nz^{\lambda_n}$, where $0<\lambda_1<\lambda_2<\cdots<\lambda_n$ and all $a_j\neq 0$. They obtain new regions in which $P(z)\neq 0$; their conditions are too complicated to state here. *A. Edrei* (Syracuse, N.Y.).

Zervos, Spiros. Une méthode de minoration des valeurs absolues des zéros des séries de Taylor. C. R. Acad. Sci. Paris **245** (1957), 394–396.

If $F(z)=\sum_0^\infty a_kz^k$ for $|z|<r$, with $a_0\neq 0$, and if ρ is the smallest positive zero (unique, when it exists) of $s(y)=|a_0|-\sum_1^\infty |a_k|y^k$, then it is clear that ρ is a minorant of the moduli of the zeros of F, and the classical minorants of the latter are usually given in terms of minorants of ρ. The author has made the astute observation that these classical minorants can all be obtained from the inequality

$$(*)\qquad 1/\rho\leqq\max\{M, [\sum_1^\infty|a_k|M_k^{t-k}/|a_0|]^{1/t}\},$$

in which t is a fixed positive number, $\{1/M_k\}_1^\infty\subset(0,r)$, and $M=\sup_k M_k<\infty$. He deduces this inequality from the following result (Lemma 2): If: (i) J is an indexing set; (ii) $\{\phi_j; j\in J\}$ is a family of non-decreasing [non-increasing] functions on some open interval I to a second open interval i; (iii) f is an extended-real-valued function on a class of sets $\{y_j; j\in J\}\subset i$, monotone in each y_j, $j\in J$, in the sense opposite to that of ϕ_j; (iv) $\{M_j; j\in J\}\subset I$; (v) $\{\phi_j(M_j); j\in J\}\subset$ domain f, (vi) $\{\phi_j(x); j\in J\}\subset$ domain f for each $x\in I$; (vii) $\mu=\inf\{M_j; j\in J\}$ and $M=\sup\{M_j; j\in J\}$ are finite; (viii) $g(x)=x^t-f(\{\phi_j(x); j\in J\})$ for a fixed $t>0$ (if t is not an integer, I should consist only of nonnegative numbers); and (ix) $x_0\in I$; then

$$\min\{\mu, [g(x_0)+f(\{\phi_j(M_j); j\in J\})]^{1/t}\}\leqq x_0$$
$$\leqq\max\{M, g[x_0)+f(\{\phi_j(M_j); j\in J\})]^{1/t}\}.$$

The inequality (*) arises by taking $g(x)=x^ts(1/x)/|a_0|$ and $x_0=1/\rho$. Various choices of $\{M_j\}_0^\infty$ give minorants which one associates with the names Cauchy, Fijiwara, Karamata, Landau, Montel, and Walsh [see, e.g., J. Dieudonné, La théorie analytique des polynomes d'une variable (à coefficientes quelconques), Mémor. Sci. Math., no. 93, Gauthier-Villars, Paris, 1938]. To mention but one such choice, if $M_j\equiv 1$, then $1/\rho\leqq\max\{1, \sum_1^\infty|a_k/a_0|\}$ [P. Montel, C. R. Soc. Sci. Lett. Varsovie. Cl. III. **24** (1932), 317–326].

A. E. Livingston (Seattle, Wash.).

Zervos, Spiros. Sur la minoration des valeurs absolues des zéros des séries de Taylor. C. R. Acad. Sci. Paris **245** (1957), 619–622.

This note continues the investigation initiated by the author in an earlier communication [see the review above]. He extends the method of that paper in two ways (the notation is that of the earlier paper): (1) If $\sum_1^\infty|a_k|y^k\leqq\tau(y)$ on $(0,\rho)$ and if $\tau(y)-|a_0|$ has a positive zero ρ_τ, then $\rho_\tau\leqq\rho$; (2) if $g(x)$ in Lemma 2 is replaced by $\alpha(u,x)=u^t-f(\{\phi_j(x); j\in J\})$, for u in a subset E of $(0,\infty)$, and if $u_0\in E$, then $x_0\leqq M$ or $u_0<[\alpha(u_0,x_0)+f(\{\phi_j(M_j); j\in J\})]^{1/t}$. Application of (1) allows him to deduce the Kuniyeda-Montel-Toya result $\rho\leqq\{1+[\sum_{k=1}^\infty|a_k/a_0|^\lambda]^{1/(\lambda-1)}\}^{(\lambda-1)/\lambda}$, $\lambda>1$, while (2) leads, for example, to a generalization of a result of Parodi [Bull. Sci. Math. (2) **79** (1955), 101–105; MR **17**, 597]. *A. E. Livingston* (Seattle, Wash.).

Pergamenceva, Ė. D. On a case of conformal mapping of a quadrilateral bounded by arcs of circles. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 2(74), 159–168. (Russian)

V. Fock [J. Reine Angew. Math. 161 (1929), 137–151] introduced a method for determining explicitly the conformal map on a half-plane of a quadrilateral bounded by arcs of circles with zero angles. Following Fock's method, the author solves the analogous problem of mapping conformally on a half-plane a quadrilateral bounded by arcs of circles, all of whose angles are equal to π, and such that two of the opposite sides, when extended, are tangent to one another. The problem reduces to the determination of a periodic integral of Lamé's differential equation. The exact solution is then used to derive simple approximate formulas of considerable accuracy. *W. Seidel.*

Komatu, Yûsaku. On conformal mapping of a domain with convex or star-like boundary. Kōdai Math. Sem. Rep. 9 (1957), 105–139.

Let E be the unit circle $|z|<1$, and R_q be the annulus $0<q<|z|<1$. Suppose $f_c(z)$ maps R_q univalently onto a ring domain such that the image of $|z|=q$ is a convex curve lying interior to the image of $|z|=1$, which is also convex, and $f_s(z)$ maps R_q univalently onto a starlike domain. The author again [Proc. Imp. Acad. Tokyo 20 (1944), 536–541; MR 7, 287] uses a Herglotz type integral representation, in a slightly different manner, to obtain known extremal properties of functions mapping E univalently onto convex or starlike domains. In the case of R_q, using a Villat-Stieltjes integral representation which involves Weierstrass elliptic functions, he gives integral representations for $1+zf_c''(z)/f_c'(z)$ and $zf_s'(z)/f_s(z)$ similar to those previously obtained by the author [Jap. J. Math. 19 (1945), 203–215; MR 7, 287]. From these representations, he finds representations for $f_s(z)$ and $f_c(z)$. Upper and lower bounds for $1+\Re\{zf_c''(z)/f_c'(z)\}$, $|f_c'(|z|)|$ (sharp only in the limit as $q\to 0$), $\Re\{zf_s'(z)/f_s(z)\}$ and $|f_s(|z|)|$ are found.

The author is evidently unaware of the work of Zmorovic [Mat. Sb. N.S. 32(74) (1953), 633–652; Dokl. Akad. Nauk SSSR (N.S.) 86 (1952), 465–468; MR 14, 1075; 15, 207], dealing with integral representations of starlike and convex functions univalent in R_q. *W. C. Royster.*

Heins, Maurice. A theorem concerning the existence of deformable conformal maps. Ann. of Math. (2) 67 (1958), 42–44.

A Riemann surface F is said to admit deformable conformal maps into itself if there is a continuous map f from $F\times[0,1]$ into F such that for each $t\in[0,1]$ the mapping $f_t:p\to f(p,t)$ is a (directly) conformal (not necessarily univalent) map of F into itself and $f_0\neq f_1$. The author proves that a hyperbolic Riemann surface F admits deformable conformal maps onto itself if and only if F admits non-constant bounded analytic functions. *H. L. Royden* (Stanford, Calif.).

Seibert, Peter. Typus und topologische Randstruktur einfach-zusammenhängender Riemannscher Flächen. Ann. Acad. Sci. Fenn. Ser. A. I. no. 250/34 (1958), 11 pp.

By a "concrete" Riemann surface over the w-plane, we mean an analytic function $w=f(z)$ defined on the region $G=\{z:|z|<R\leq\infty\}$. The author introduces a metric on the surface (i.e., on G) by setting

$$\rho(z_1,z_2)=\max_{i=1,2}\inf_C\max_{z\in C}[f(z),f(z_i)],$$

where C rangesover all Jordan curves in G joining z_1 to z_2, and $[w_1,w_2]$ is the spherical distance from w_1 to w_2. The boundary Z of the surface is taken to be the set of equivalence classes of non-convergent Cauchy sequences. If $\zeta=\{z_\nu\}$ is such a Cauchy sequence, then $f(z_\nu)$ converges to some point a, and we say that ζ lies over a. The set of $\zeta\in Z$ which lie over a given point a is denoted by Z_a. The author shows that the cyclic order of the boundary of G induces a cyclic order in Z.

The author proves the following theorems. Theorem 2: Let Λ be a complete separable metric space with a cyclic order. To each ordered pair $\langle\alpha,\beta\rangle$ of elements of Λ, let there be a positive number $\rho^*(\alpha,\beta)$, defined such that $\rho^*(\alpha,\beta)=\max[\rho^*(\alpha,\gamma),\rho^*(\gamma,\beta)]$, for all γ such that $\alpha\gamma\beta$ are in the cyclic order, and such that the metric $\rho(\alpha,\beta)=\min[\rho^*(\alpha,\beta),\rho^*(\beta,\alpha)]$. Then there are (concrete) Riemann surfaces of given type (i.e. parabolic or hyperbolic), such that $Z=Z_0\cong\Lambda$. It should be noted that each discontinuum Λ of the unit circumference satisfies the hypothesis of this theorem. Theorem 3. To each closed subset Λ of the unit circumference, there exist concrete Riemann surfaces of both types such that $Z\cong\Lambda$. *H. L. Royden.*

Wilson, R. The directions of strongest growth of an integral function of finite order and mean type. J. London Math. Soc. 32 (1957), 409–420.

Let $F(z)$ be entire, of order ρ and type h, $0<\rho<\infty$, $0<h<\infty$, and let $h(\theta)=\lim\sup_{r\to\infty}r^{-\rho}\log|F(re^{i\theta})|$. Then the values of θ for which $h(\theta)=h$ are called the directions of strongest growth for F. If θ_0 is one of these, then for any $h'<h$, the inequality

$$(*)\qquad |F(re^{i\theta_0})|>\exp(h'r^\rho)$$

holds for a set of values of r of positive upper linear density. The author is concerned with refinements of this type of result. The key is the use of Macintyre's generalized Borel-Laplace transform [Proc. London Math. Soc. (2) 45 (1938), 1–20], which associates with the function F a unique function f (which can also be obtained from the power series for F), such that $f(z)$ is analytic at least outside the circle $|z|=h^\sigma$, where $\sigma=\rho^{-1}$. The directions of strongest growth for F correspond to the positions of singularities of f on this circumference, and the linear density of the set of r values for which (*) holds is conditioned by the nature of these singularities and their relationship to neighbouring singularities of f. This approach is an outgrowth of the fundamental work of Pólya [Math. Z. 29 (1929), 549–640; Ann. of Math. (2) 34 (1933), 731–777; see also Bieberbach, Analytische Fortsetzung, Springer, Berlin, 1955; MR 16, 913; and R. Wilson, J. London Math. Soc. 28 (1953), 185–193; MR 14, 739]. The singularities of f (and the directions of strongest growth of F) are classified in a detailed way ("unique", "virtually isolated", "easily approachable"), so that quite precise results can be obtained, of which the following is typical. Theorem: If F has only one direction of strongest growth, and it is either isolated, accessible, or unique, then the inequality (*) holds for a set of r values which has linear density 1, or upper linear density 1, or maximal (outer) density 1, respectively. Other theorems of a similar nature are obtained which deal with the behavior of functions F having several, or even a whole sector, of directions of strongest growth. *R. C. Buck.*

Wilson, R. Hadamard multiplication of integral functions of finite order and mean type. J. London Math. Soc. 32 (1957), 421–429.

Denote the Hadamard product of F_1 and F_2 by $F_1\odot F_2$.

Much is known about the connection between singularities of the factor functions and those of the product. [See Bieberbach, Analytische Fortsetzung, Springer, Berlin, 1955; MR 16, 913.] In the extreme case when F_1 and F_2 are entire, of order ρ_j, type h_j, then $F_1 \odot F_2$ is entire, and its order ρ and type h are limited by the inequalities (where $\sigma_j = 1/\rho_j$)

$$(*) \qquad \rho \leq \rho_1\rho_2/(\rho_1+\rho_2),$$

$$(**) \qquad h \leq (\sigma_1+\sigma_2)\exp\left\{\frac{\sigma_1 \log(h_1/\sigma_1)+\sigma_2 \log(h_2/\sigma_2)}{\sigma_1+\sigma_2}\right\}.$$

It is natural, in this case, to look for connections between the directions of strongest growth of the factor functions, and those of the product. Let f_j be the Macintyre-Borel transform of F_j [Macintyre, Proc. London Math. Soc. (2) **45** (1938), 1–20]. As in the paper reviewed above, the directions of strongest growth of an entire function can be studied by means of the singularities of its transform. If f is the transform of $F_1 \odot F_2$, then $f = f_1 \odot f_2 \odot B$, where

$$B(z) = \sum_0^\infty \frac{\Gamma([n+1](\sigma_1+\sigma_2))z^{-n-1}}{\Gamma([n+1]\sigma_1)\Gamma([n+1]\sigma_2)}.$$

Suppose now that F_1 has only one direction θ_1 of strongest growth, and that θ_2 is one of the directions for F_2. It is reasonable to hope that $F_1 \odot F_2$ will have $\theta_1+\theta_2$ as a direction of strongest growth, and that its order and type will be given by taking equality in (*) and (**). The author, using results in the paper reviewed above, shows that this is the case in three more or less well behaved cases: (i) if F_1 is the generalized exponential function E_ρ; (ii) if θ_1 is an isolated direction for F_1, and θ_2 is not an interior direction for a whole arc of extremal directions; (iii) if θ_1 is an accessible direction for F_1 and θ_2 is an isolable direction for F_2. By splitting the functions, these results can be suitably modified to extend to functions F_1 having several directions of strongest growth.

R. C. Buck (Madison, Wis.).

Goldberg, A. A. On an inequality connected with logarithmic convex functions. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 227–230. (Ukrainian. Russian and English summaries)

For $1 \leq \nu \leq n$, let $y_\nu(x)$ be even, bounded, with period 2π, and monotone increasing for $0 \leq x \leq \pi$; and let $\Phi(y)$ be logarithmically convex. Then the following inequality is proved:

$$\int_{-\pi}^{\pi} \Phi\left(\prod_{\nu=1}^{n} y_\nu(x+\tau_\nu)\right)dx \leq \int_{-\pi}^{\pi} \Phi\left(\prod_{\nu=1}^{n} y_\nu(x)\right)dx.$$

From it a number of new inequalities can be obtained for meromorphic functions, an instance of which is the following. Let $f(z)$ be a function meromorphic in the finite z-plane of genus zero and let $\mathring{f}(z)$ also be a meromorphic function of genus zero, $f(0) = \mathring{f}(0) \neq 0$, whose moduli of zeros and poles are respectively equal to the moduli of zeros and poles of $f(z)$, but all the zeros of which are distributed along the positive real axis and all the poles along the negative real axis. Then

$$\int_0^{2\pi} |f(re^{i\theta})|^\alpha d\theta \leq \int_0^{2\pi} |\mathring{f}(re^{i\theta})|^\alpha d\theta$$

is valid for all r, where $0 \leq r < \infty$, and every α, where $-\infty < \alpha < +\infty$. *Author's summary.*

★ **Hiong, King-Lai. Sur les fonctions méromorphes et les fonctions algébroïdes. Extensions d'un théorème de M. R. Nevanlinna.** Mémor. Sci. Math., no. 139. Gauthier-Villars, Paris, 1957. 104 pp.

In this monograph, classical problems of Nevanlinna's value distribution theory for meromorphic functions are treated by means of purely analytical tools. It is assumed that the reader possesses a certain acquaintance with the problems and methods of this theory.

Applying the standard methods of Nevanlinna theory and making use of results by Montel, Miranda, Milloux and others, the author studies, in the first chapters, relations between the distribution of values of a meromorphic function and its derivatives. Because of the character of the problems posed, the results in this direction are often rather incomplete and formally complicated. In spite of this, the author has succeeded in finding some interesting new results [Bull. Sci. Math. (2) **80** (1956), 175–190; MR **19**, 129].

After this, value distribution properties of algebroid functions are dealt with. More than twenty years have now passed since a fairly systematic theory was built up, primarily by H. Selberg and Valiron, but no unified representation has been published before. It is, therefore, well motivated to include this theory in the book, although this chapter contains little new material.

The last part of the monograph is devoted to the study of linear combinations of analytic functions. H. and J. Weyl's and Ahlfors' theory of meromorphic curves is merely mentioned; the chief attention is concentrated on the theory developed by H. Cartan, which is reviewed in fairly great detail. *O. Lehto* (Helsinki).

Lohwater, A. J.; and Piranian, G. The boundary behavior of functions analytic in a disk. Ann. Acad. Sci. Fenn. Ser. A. I. no. **239** (1957), 17 pp.

Let R be the class of functions $f(z)$ regular in the unit disc D, $|z|<1$, of boundary C, for which $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists for all θ. If in addition $f(z)$ is not a constant and $|f(e^{i\theta})|$ is either 0 or 1 the class will be called R_0. Let A be the class of functions $f(z)$ regular and bounded in D and for which the radial limit $f(e^{i\theta})$ has modulus 1 for almost all θ. The following results are obtained.

In order that a set E on C be the set of discontinuities of $f(e^{i\theta})$ for $f(z)$ in R, it is necessary and sufficient that it be of type F_σ (in the sense of Baire) and of first category. If $f(z)$ is further required to be univalent in D no additional restriction is put on the set E.

In order that a set E on C be the set where $f(e^{i\theta}) = 0$ for some function $f(z)$ of R_0 it is necessary and sufficient that E be of measure zero and simultaneously of types F_σ and G_δ.

If the set E on C has measure zero there exists a function of A whose modulus oscillates between 0 and 1 on every radius of D terminating in the set E. If E is of types F_σ and G_δ and of measure zero, there exists a function $f(z)$ of A whose modulus oscillates between 0 and 1 on every radius of D terminating in the set E, and for which on each point $e^{i\theta}$ of $C-E$ the radial limit $f(e^{i\theta})$ exists and has modulus 1, except for a denumerable set of points where $f(e^{i\theta}) = 0$.

Let Γ be a simple closed Jordan curve, internally tangent to C at $z=1$, and having no other points on C; and let Γ_θ denote the image of Γ under a rotation through an angle θ about the origin. Then there exists a function $f(z)$ of A which does not approach a limit as z approaches any point $e^{i\theta}$, from left or from the right, along Γ_θ.

There exists a function $f(z)$, meromorphic in D, with $T(r, f) = O(\log(1-r)^{-1})$, and such that $f(e^{i\theta})$ does not exist for any value of θ.

M. S. Robertson.

Miki, Yoshikazu. A note on close-to-convex functions. J. Math. Soc. Japan 8 (1956), 256–268.

Let $f(z)=z+a_2z^2+\cdots$ and $g(z)=z+b_2z^2+\cdots$ be analytic for $|z|<1$. Let $g(z)$ be convex and let $\mathrm{Re}(f'/g')>0$ for $|z|<1$, so that f is schlicht and close-to-convex for $|z|<1$, as defined by the reviewer [Michigan Math. J. 1 (1952), 169–185; MR 14, 966]. Let $f_n(z)$, $g_n(z)$ be the nth partial sums of the series for f and g. The author proves that $\mathrm{Re}(f_n'/g_n')>0$ for $|z|<\frac{1}{4}$ and that $\frac{1}{4}$ is the best possible constant; this theorem implies that f_n is also close-to-convex for $|z|<\frac{1}{4}$ and is hence a generalization of a theorem of Szegö [Math. Ann. 100 (1928), 188–211]. The reviewer remarks that the result applies only to a subclass of the class of close-to-convex functions, since $g(z)$ is assumed normalized. *W. Kaplan* (Ann Arbor, Mich.).

Lebedev, N. A. On the domains of values of a certain functional in the problem of non-overlapping domains. Dokl. Akad. Nauk SSSR (N.S.) 115 (1957), 1070–1073. (Russian)

Let $w=f(z)$, $f(0)=0$, be regular and univalent in $|z|<1$; and let $w=F(\zeta)$, $F(\infty)=\infty$, be regular and univalent in $1<|\zeta|<\infty$. If the image regions under f and F have no point in common, the pair (f, F) is said to belong to the class $\mathfrak{M}$. The author determines the domain E of values for the ratio $f(z_0)/F(\zeta_0)$ for fixed z_0, ζ_0 as the pair (f, F) range through the class $\mathfrak{M}$.

The author uses variational methods and only an outline of the proof is given. The formula which he obtains for the equation of the boundary of the set E is too complicated to be reproduced here. *A. W. Goodman.*

Okuda, Hidesuke; and Sakai, Eiichi. On the continuation theorem of Levi and the radius of meromorphy. Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math. 11 (1957), 65–73.

The author gives a new proof of Kneser's generalization of Levi's continuity theorem in the meromorphic, circular case and for n variables. He further proves that this case of the continuity theorem is equivalent to the fact that the logarithm of the radius of meromorphy is superharmonic. This holds whether the radius of meromorphy is understood in the classical sense (Hartog's) or in the sense of Rothstein [Math. Z. 53 (1950), 84–95, p. 86; MR 12, 252]. *H. Tornehave* (Copenhagen).

Sakai, Eiichi. A note on meromorphic functions in several complex variables. Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math. 11 (1957), 75–80.

The author proves the following generalization of Hartog's classical theorem: Let C denote the polycylinder $|z_\nu|<1$, $\nu=1, \cdots, n$, and let $E\subset C$ be an exceptional point set which is closed and nowhere dense in C and which does not divide any subdomain of C. Let $f(z_1, \cdots, z_n)$ be defined on $C-E$ and in the vicinity of every point of $C-E$ meromorphic in each variable for fixed values of the remaining variables. Then f is meromorphic in C. The proof is based on results by W. Rothstein [Math. Z. 53 (1950), 84–95; MR 12, 252]. *H. Tornehave* (Copenhagen).

Ono, Isao. Analytic vector functions of several complex variables. J. Math. Soc. Japan 8 (1956), 216–246.

The author has proved theorems of the following kind. In k complex variables, if for a holomorphic mapping $w_j(z)=z_j+$(higher powers), $j=1, \cdots, k$, one has $\|w(z)\|<1$ for $\|z\|<1$, then the mapping is univalent in $\|z\|<(2(k+1)M)^{-1}$ and contains a univalent sphere of radius $(4(k+1)M)^{-1}$. For a nonspherical domain he has, for the number of images $n(0)$ of any holomorphic mapping, an expression which is the sum of a boundary integral and a volume integral from which he draws conclusions. Technically, when expanding a holomorphic function around the origin, the author does not use the monomials $z_1^{n_1}, \cdots, z_k^{n_k}$ but he multiplies them by the square root of $\dfrac{(n_1+\cdots+n_k)!}{n_1!\cdots n_k!}$, and he has a related technique for expressing the Taylor series of a holomorphic mapping. *S. Bochner* (Princeton, N.J.).

Rothstein, Wolfgang. Zur Theorie der analytischen Mannigfaltigkeiten im Raume von n komplexen Veränderlichen. Die Fortsetzung analytischer Mengen in Gebieten mit analytischen Schlitzen. Math. Ann. 133 (1957), 400–409.

In his previous papers [Math. Ann. 129 (1955), 96–138; 133 (1957), 271–280; MR 17, 84; 19, 766], the author has shown results of the following type about the continuation of analytic sets: given two bounded domains G and G_0, $G_0\Subset G$, in the complex number space C^n; let $\mathfrak{g}^k$ be an analytic set in $G-G_0$; then under various conditions on G (r-convexity) and on the dimension k, there exist an analytic set $\mathfrak{g}_*^k$ in G and a domain $G_*\Subset G$ such that $\mathfrak{g}_*^k$ coincides with $\mathfrak{g}^k$ in $G-G_*$. It is shown that all results on the continuation of analytic sets of the papers quoted remain valid if the domains are replaced by domains "slit" with a generalized analytic set M; that is, if in the above statement G, G_0, G_*, $G-G_0$, and $G-G_*$ are replaced by $G-M$, G_0-M, G_*-M, $G-G_0-M$, and $G-G_*-M$ respectively where M is an "F-set". M is an "F-set" if it is locally of the form $(f_1-c_1=0, \cdots, f_l-c_l=0)$, $1\leq l\leq n$, where $f_1, \cdots, f_l$ are holomorphic and $c_1, \cdots, c_l$ vary independently in sets $E_1, \cdots, E_l$ (of the plane) of harmonic measure zero. In particular, analytic sets are F-sets. *H. J. Bremermann* (Seattle, Wash.).

See also: Partial Differential Equations: Dressel and Gergen; Migliau. **Elasticity, Plasticity:** Mişicu.

Harmonic Functions, Convex Functions

Hunt, G. A. Markoff processes and potentials. I, II. Illinois J. Math. 1 (1957), 44–93, 316–369.

Cet article très important et original établit des connexions étroites entre la théorie générale du potentiel et les processus de Markov homogènes. On sait depuis longtemps déjà les relations entre la théorie newtonienne et le mouvement brownien [voir notamment J. L. Doob, Trans. Amer. Math. Soc. 77 (1954), 86–121; MR 16, 269]; le cas traité ici est bien plus général et plus délicat. L'idée essentielle est la suivante: moyennant certaines hypothèses de régularité sur lesquelles nous allons donner quelques détails, si les P_t sont les probabilités de passage d'un processus de Markov homogène, le "noyau" $U=\int_0^\infty P_t dt$ satisfait aux principes fondamentaux de la théorie du potentiel, tels que les principes du maximum; inversement à un noyau satisfaisant à ces principes correspond un semi-groupe de probabilités de passage. Cela conduira à des interprétations probabilistes simples des énoncés de la théorie générale du potentiel, et fournira un puissant outil de démonstration, permettant même d'atteindre la théorie "fine".

Soit $\mathscr{X}$ un espace localement compact séparable. L'auteur appelle noyau sur $\mathscr{X}$ toute famille $\{K_r\}_{r\in\mathscr{X}}$ de mesures de Radon positives sur $\mathscr{X}$, telle que la charge $K(r, A)$ de K_r sur le borélien $A\subset\mathscr{X}$ soit une fonction de r mesurable pour toute mesure de Radon. Un tel noyan

définit une transformation $f\to Kf$ des fonctions positives mesurables par la formule $Kf(r)=\int K(r, ds)f(s)$ et, sous certaines conditions de finitude, une transformation $\mu\to\mu K$ des mesures positives par la formule $\mu K(A)=\int \mu(dr)K(r, A)$. On définit de manière évidente le produit HK de deux noyaux K et H.

On part d'un semi-groupe de noyaux $P_t(r, A)$ $(0\leqq t<\infty)$ sur $\mathscr{H}$, les mesures définissant chacun d'eux étant des mesures de probabilité $(P_t(r, \mathscr{H})=1)$. On suppose que pour toute mesure de probabilité μ sur $\mathscr{H}$ il existe un processus de Markov homogène $X(\omega, t):\Omega\times[0, \infty)\to\mathscr{H}$ qui admet μ pour loi initiale, les P_t pour probabilités de passage, et qui est régulier au sens suivant: 1°) les chemins aléatoires $X(\omega, \cdot)$ sont continus à droite et ont des limites à gauche; 2°) la propriété de Markov "étendue" a lieu [voir R. Blumenthal, ibid. **85** (1957), 52–72; MR **19**, 468]; 3°) si $\{T_n\}$ est une suite croissante de "temps de Markov" [cf. Blumenthal, loc. cit.], $\lim X(\omega, T_n(\omega))=X(\omega, \lim T_n(\omega))$ pour presque tout ω pour lequel les $T_n(\omega)$ sont bornés. Il résulte des travaux de Blumenthal [loc. cit] et de Kinney [ibid. **74** (1953), 280–302; MR **14**, 772] que ces trois conditions sont réalisées dans le cas important où les opérateurs $P_t: f\to P_tf$ laissent invariants l'espace des fonctions continues sur $\mathscr{H}$ tendant vers 0 à l'infini, et constituent un semi-groupe fortement continu.

On considère le noyau $U(r, A)=\int_0^\infty P_t(r, A)dt$, l'intégrale étant supposée convergente pour tout borélien borné A. On appelle Uf potentiel de la fonction f, μU potentiel de la mesure μ. Lorsque $\mathscr{H}=R^3$ et que les P_t sont les opérateurs de convolution par les distributions de Gauss, U est le noyau newtonien $|r|^{-1}$ (plus exactement $U(r, A)=\int_A |r-s|^{-1}ds$); Uf est le potentiel newtonien engendré par f, μU est la mesure ayant pour densité le potentiel newtonien engendré par μ.

Si $X_r(\omega, t)$ est un processus régulier "partant" de r $(X_r(\omega, 0)=r)$ et admettant les P_t pour probabilités de passage, on a la formule fondamentale

$$Uf(r)=\int_\Omega d\omega\int_0^\infty f(X_r(\omega, t))dt$$

d'où l'on déduit aisément les énoncés fondamentaux de la théorie du potentiel: principes du maximum pour les potentiels Uf, principe du balayage pour les potentiels μU. A signaler à ce propos que la dualité entre ces principes a été établie indépendamment, d'une façon entièrement différente, par G. Choquet et le reviewer [C. R. Acad. Sci. Paris **243** (1956), 764–767; MR **19**, 848].

Il y a deux sortes de généralisations de la notion de fonction surharmonique positive: les fonctions et les mesures "excessives". Une fonction $\varphi\geqq0$ est dite excessive si elle est mesurable pour toute mesure de Radon et si $P_t\varphi$ tend partout vers φ en croissant lorsque t décroit vers 0. Exemples: les constantes positives, les potentiels Uf, la probabilité $\Phi_E(r)$ pour que X_r rencontre l'ensemble analytique E (généralisation de la notion de potentiel d'équilibre). On étend à ces fonctions les propriétés connues concernant les enveloppes de familles de fonctions surharmoniques; d'autre part si φ est excessive et X un processus régulier associé aux P_t, $\varphi(X(\omega, t))$ est une semi-martingale au sens de Doob [loc. cit.] pourvu que l'espérance mathématique de $\varphi(X(\omega, 0))$ soit finie. Une mesure $\zeta\geqq0$ est excessive si $\zeta P_t\leqq\zeta$ pour tout $t\geqq0$. Moyennant une hypothèse de régularité supplémentaire, on démontre un théorème de décomposition de Riesz: toute mesure excessive est de la forme $\xi+\mu U$, avec $\mu\geqq0$ et ξ excessive, analogue aux fonctions harmoniques usuelles en ce sens que $\xi=L\xi$, L étant l'opérateur de "balayage" sur le complémentaire d'un ouvert relativement compact quelconque.

La variable aléatoire $T_r(\omega)=\inf\{t>0;\ X_r(\omega, t)\in E\}$, où E est analytique, s'annule avec probabilité 0 ou 1, d'après la propriété de Markov étendue. Selon le cas, le point r sera dit régulier ou irrégulier pour E. Posons $H_E(r, A)=\text{prob}\{T_r(\omega)<\infty;\ X_r(\omega, T_r(\omega))\in A\}$; si φ est excessive $H_E\varphi$ est excessive et on a $H_E\varphi\leqq\varphi$ partout, $H_E\varphi=\varphi$ en tout point régulier de E. L'opération $\varphi\to H_E\varphi$ généralise l'extrémisation des fonctions surharmoniques selon Brelot [J. Math. Pures Appl. (9) **24** (1945), 1–32; MR **7**, 521]. Le noyau H_E peut être interprété comme la généralisation de la notion classique de masses de Green; il est utile dans la définition précise du balayage. Tout ceci est le point de départ de la théorie fine du potentiel.

En réalité, les P_t étant donnés, on considère non seulement le noyau U, mais toute une famille de noyaux, associés à divers systèmes de "temps d'arrêt" (terminal times). Par exemple, si R_r est le temps où le processus X_r rencontre le borélien $B\subset\mathscr{H}$, on pose $K_t(r, A)=\text{prob}\{X_r(\omega, t)\in A;\ R_r(\omega)>t\}$; le noyau $V(r, A)=\int_0^\infty K_t(r, A)dt$ est l'analogue de la fonction de Green du complémentaire de B. La formule fondamentale pour les potentiels est alors

$$Vf(r)=\int_\Omega d\omega\int_0^{R_r(\omega)} f(X_r(\omega, t))dt.$$

Un autre exemple est constitué par les noyaux $U^\lambda(r, A)=\int_0^\infty e^{-\lambda t}P_t(r, A)dt$ avec $\lambda>0$. D'autres exemples sont nouveaux, même dans le cas newtonien; on démontre pour ces noyaux des théorèmes analogues à ceux qui ont été énoncés pour U: c'est la "théorie relative".

Le point de vue inverse est étudié vers la fin de la seconde partie; on part d'un noyau positif V sur un espace localement compact $\mathscr{X}$, appliquant l'ensemble $\mathscr{B}$ des fonctions continues à support compact dans l'ensemble $\mathscr{C}$ des fonctions continues tendant vers 0 à l'infini. On suppose en outre: (i) l'image de $\mathscr{B}$ par V est dense dans $\mathscr{C}$; (ii) si a est une constante $\geqq0$, et si f et g sont deux fonctions positives de $\mathscr{B}$, la relation $Vf\leqq a+Vg$ a lieu partout si elle a lieu sur le support de f (principe complet du maximum).

Alors on prouve l'existence d'un semi-groupe fortement continu d'opérateurs "sous-stochastiques" K_t tels que $V=\int_0^\infty K_tdt$. En ajoutant au besoin un point w à l'espace $\mathscr{X}$, on peut prolonger les K_t à $\mathscr{H}=\mathscr{X}\cup\{w\}$ de façon à obtenir un semi-groupe d'opérateurs P_t satisfaisant à $P_t(r, \mathscr{H})=1$. Le noyau V se présente donc comme le noyau de Green relatif au sous-espace $\mathscr{X}$ de $\mathscr{H}$ dans la théorie construite à partir des P_t. Ceci résout l'un des problèmes fondamentaux de la théorie du potentiel.

Ce travail constitue donc un progrès essentiel dans l'étude des aspects "linéaires" de la théorie du potentiel; par contre les questions d'énergie (lorsque le noyau présente certains caractères de symétrie) et les problèmes aux limites y sont délibérément laissés de côté. La lecture n'en est pas facile. Cela tient peut-être en partie à ce que les définitions et hypothèses ne sont pas toujours clairement mises en évidence, et que, malgré la longueur du texte, certaines démonstrations sont seulement esquissées; cela tient surtout à ce que les préliminaires sont longs et font appel à beaucoup de connaissances et à des résultats récents et difficiles tels que les travaux déjà cités de Blumenthal et Kinney, et la théorie des fonctions d'ensembles alternées d'ordre infini, dûe à Choquet [Ann. Inst. Fourier, Grenoble **5** (1953–1954), 131–295; MR **18**, 295]; les questions de mesurabilité, notamment, posent

des problèmes délicats. Le lecteur qui aura fait l'effort de comprendre ces préliminaires pourra apprécier l'élégance avec laquelle la théorie générale s'en déduit, et la fécondité des méthodes utilisées. *J. Deny* (Strasbourg).

Matsushita, Shin-ichi. Generalized Laplacian and balayage theory. J. Inst. Polytech. Osaka City Univ. Ser. A. **8** (1957), 57–90.

This paper is a very general postulational investigation of certain aspects of potential theory. The author assumes a transformation φ from measures to functions which is positive with respect to a suitable cone in the function space, and a positive transformation Δ from functions to measures which is defined at least on the range of φ and inverts φ there. In this setting the author discusses such topics as the Riesz decomposition of functions, in the domain of Δ, and equilibrium distributions, together with the associated notion of capacity. Additional assumptions, such as a form of the maximum principle and the positivity of the energy integral $\int \varphi(\mu)d\mu$, are made as needed, but the preliminary investigation proceeds without them, and covers, for example, the Bochner mapping from positive definite functions to their associated measures on Euclidean spaces. There is no discussion of what properties of a kernel imply the maximum principle; the latter is simply assumed from the classical theories in the various special cases where it holds. The proof that the maximum principle implies the possibility of balayaging measures is here made to depend on the Krein-Milman theorem. This method has the advantage of applying directly to certain measures of infinite energy which ordinarily are treated as limiting cases of the finite energy situation. The paper concludes with a discussion of a general Dirichlet problem for continuous boundary values. *L. H. Loomis* (Cambridge, Mass.).

Kozmanova, A. A. Deduction of the inverse problem equation in the theory of Newton's potential. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 21–23. (Russian)

The problem is that of finding a simply connected region D with a given real constant density when an external potential is given. It is solved in E_3 by finding a vector function $\mathbf{r}(x, y, z)$ which maps the unit sphere in E_3 onto D. Based on the assumption that $\mathbf{r}$ is "potential-harmonic" (i.e., has vanishing divergence and curl) outside this sphere, the author finds a non-linear integro-differential equation satisfied by $\mathbf{r}$. The result extends to three dimensions an earlier work of V. K. Ivanov [same Dokl. (N.S.) **105** (1955), 409–411; MR **17**, 1196] which solved the same problem in E_2. *R. N. Goss.*

See also: Functions of Complex Variables: Goldberg. **Partial Differential Equations:** Migliau; Laasonen.

Special Functions

Newman, Morris. Construction and application of a class of modular functions. Proc. London Math. Soc. (3) **7** (1957), 334–350.

The author considers functions defined for $\Gamma_0(n)$ a subgroup of the modular group in the τ plane. (Here $c\equiv 0 \bmod n$ in the usual $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ notation, and $(n, 6)=1$.) In particular, G_n is the (multiplicative) subgroup of the group generated by $\eta(\delta\tau)/\eta(\tau)$ where $\delta|n$; H_n is the subset of G_n with negative valences only at $i\infty$; F_n the entire functions meromorphic at parabolic points; and finally E_n is the totality of F_n with negative valences only at $i\infty$. Let ν be the genus of $\Gamma_0(n)$. Then a polynomial basis of E_n is known to have $\nu+1$ elements by the Riemann-Roch theorem. Since $E_n \supset H_n$ the author would like to choose the polynomial basis of E_n in H_n. This cannot be done generally since when n is a prime the elements of H_n are powers of a single function while the genus is seldom 0; but the conjecture is made that for n composite the basis can be so chosen.

Thus a tremendous advantage accrues from the fact that G_n and hence H_n can be explicitly characterized using the coefficients of the power series expansion in $\exp 2\pi i\tau$. The author considers the case of $n=35$ where $\nu=3$. The basis of E_{35} consists of four functions of valences -4, -5, -6, -7 built with linear combinations of nine functions of G_{35} of which a typical example is $\eta(5\tau)^6\eta(7\tau)/\eta(\tau)\eta(35\tau)^6$. A polynomial basis of F_{35} is also constructed.

The SEAC electronic computer was used to calculate power series coefficients and to reduce the matrices used in determining the aforementioned linear combinations. (The series-length and matrix-size were known in advance.) *Harvey Cohn* (Tucson, Ariz.).

Herrmann, Oskar. Über den Rang der Schar der Spitzenformen zu Hilbertschen Modulgruppen beliebiger totalreeller Körper. Arch. Math. **8** (1957), 322–326.

The author extends his work on the existence of non-vanishing cusp forms [Math. Z. **64** (1956), 457–466; MR **17**, 1059] to an arbitrary totally real field of degree n. He shows in fact that the rank r of the cusp forms of dimension $-k$ of the Hilbert modular group of a fixed field satisfies:

$$\liminf_{k\to\infty} \frac{r}{k^n} \geq \frac{R|d|^{\frac{1}{2}}}{2(\pi e)^n},$$

where R is the regulator, d is the discriminant and only those k are considered for which nk is even, the other ranks being trivially zero.

The special device is the Poincaré series in τ

$$G_\mu = \sum_{(\gamma,\delta)} \sum_{\lambda} \mathrm{N}(\gamma\tau+\delta)^{-k} \exp 2\pi i \mathrm{S}\left(\frac{\mu\lambda^2}{\partial}\Gamma\tau\right),$$

where μ is totally positive, and $\Gamma\tau$ runs over the group matrices with $(\gamma\tau+\delta)$ as denominator while λ runs over the units mod ± 1. The different, ∂, is assumed to be principal, for convenience. Then in the Fourier series,

$$G_\mu = \sum_{\nu} a_{\mu\nu} \exp 2\pi i\ \mathrm{S}\left(\frac{\nu\tau}{\partial}\right),$$

$a_{\mu\nu}$ can be expanded into a sum of the type: $\delta_{\mu\nu}+$"small terms" (i.e., combinations of Kloosterman sums and Bessel functions), while from now on μ and ν are specialized to a sequence of inequivalent totally positive integers ordered by norm: $\mu_1, \mu_2, \cdots, \mu_r$. The main theorem is based on an estimate of how big r can be made without disturbing the dominance of the main diagonal of determinant $|a_{\mu\nu}|$, thus ensuring linear independence of these r series G_μ. *Harvey Cohn* (Tucson, Ariz.).

Myrberg, P. J. Eine Anwendung der Differenzengleichungen auf gewisse automorphe Funktionen zweier Variablen, deren Gruppe kommutativ ist. Ann. Acad. Sci. Fenn. Ser. A. I. no. **248** (1958), 10 pp.

Previous results [same Ann. no. **235** (1957); MR **19**, 740] are extended to the case of commutative groups of transformations, $x'=\alpha_\nu x+\beta_\nu$, $y=a_\nu(x)y+b_\nu(x)$, having one or two generators. *H. S. Zuckerman.*

Al-Salam, W. A.; e Carlitz, L. A q-analog of a formula of Toscano. Boll. Un. Mat. Ital. (3) 12 (1957), 414–417.

Une formule due à Toscano, pour les polynomes H_n d'Hermite est (si $1 \leq m \leq n$):

$$\sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} H_{n+r}(x) H_{n-r}(x) = \frac{(2m)!(n-m)!}{m} \sum_{k=m}^{n} \binom{k-1}{m-1} \frac{H_{n-k}^2(x)}{(n-k)!}.$$

L'auteur démontre une formule analogue pour les polynomes

$$H_n(x, q) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^r$$

où

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}, \begin{bmatrix} n \\ 0 \end{bmatrix} = 1,$$

polynomes qui ont des propriétés voisines de celles des polynomes d'Hermite. *R. Campbell* (Caen).

★ **Robin, Louis. Fonctions sphériques de Legendre et fonctions sphéroïdales. Tome I. Préface de H. Villat.** Gauthier-Villars, Paris, 1957. xxxv+201 pp. Broché 4000 francs; cartonné 4300 francs.

Ce tome I de la théorie des fonctions sphériques comprend 3 chapitres. Le premier est consacré à l'introduction de l'équation différentielle associée de Legendre en physique mathématique, et aux polynomes de Legendre proprement dits. Il faut remercier l'auteur d'avoir, dès le début, introduit l'équation différentielle associée à partir de la séparation des variables sur un système triple orthogonal. Ainsi est-il clairement montré qu'il s'introduit deux constantes de séparation et que l'équation classique de Legendre ne s'obtient que comme un cas très particulier de celle-ci (cas où l'une de ces constantes est nulle). Aussitôt d'ailleurs, on étudie ce cas particulier avec toute l'ampleur qu'il mérite.

Tout ce qu'il faut savoir sur les polynomes P_n de Legendre et les fonctions de 2ème espèce Q_n est signalé, y compris le développement d'une fonction en série de polynomes P_n et des exemples d'intégrales où interviennent les fonctions Q_n, ainsi que des développements en séries de fonctions (analytiques) contenant des Q_n.

Le deuxième chapitre étudie les fonctions associées P_n^m et Q_n^m pour n et m entiers ($n \geq 0$). La théorie en est très clairement et complètement exposée, avec des détails tel que la définition de la fonction d'Adolf Schmidt, utilisée par les géophysiciens, des exemples d'intégrales définies contenant les Q_n^m.

Le chapitre III est consacré à l'étude des harmoniques sphériques de degré entier positif ou nul. L'auteur insiste très heureusement à cette occasion sur le lien existant entre les harmoniques sphériques et les fonctions harmoniques à deux variables.

Le volume se termine par quelques tables numériques donnant les expressions de P_n, Q_n, P_n^m, Q_n^m pour les premières valeurs de l'entier n.

Malgré la densité de sa substance, l'ouvrage n'est pas pesant; les formules ne se succèdent pas sans aération, comme il arrive souvent dans ce genre de traité. Il y a même quelques figures géométriques illustrant un peu les propriétés de ces fonctions "sphériques".

Pour notre part, il nous paraîtra toujours regrettable qu'un ouvrage relatif à des fonctions orthogonales utilise de telles fonctions sans les normer. Mais il y a, pour les polynomes de Legendre, comme pour ceux de Jacobi en général, ceux de Laguerre et de Hermite, une tradition qui veut qu'on les étudie avec des normes dépendant de l'indice et l'auteur a eu bien raison de s'y conformer, faute de quoi son livre serait devenu inutilisable. Ses notations, qui sont, au contraire, les plus habituelles, ne dépaysent absolument pas et donnent au lecteur l'impression d'une familiarité immédiate avec le texte, contribuant largement à la facilité de sa lecture.

R. Campbell (Caen).

Venkatachaliengar, K.; and Lakshmana Rao, S. K. On Turán's inequality for ultraspherical polynomials. Proc. Amer. Math. Soc. 8 (1957), 1075–1087.

The authors generalize various results, known for the Legendre polynomials $P_n(x)$, to the Gegenbauer functions $P_\alpha^{(\lambda)}(x)$ for $\lambda > 0$, $\alpha \geq 1$, x real. Most of the results are obtained for real values of α, rather than for positive integral values. In the latter case, $P_\alpha^{(\lambda)}(x)$ becomes an ultraspherical (or Gegenbauer) polynomial.

The Turán expression

$$\Delta_\alpha^{(\lambda)}(x) = [P_\alpha^{(\lambda)}(x)]^2 - P_{\alpha+1}^{(\lambda)}(x) P_{\alpha-1}^{(\lambda)}(x)$$

is under consideration. The sign of its derivative with respect to x is determined for $\lambda > 0$, $\alpha \geq 1$, and from this, monotonicity theorems are obtained. The main result is that $\Delta_\alpha^{(\lambda)}(x)$ is a concave or convex function of x, for $\frac{1}{2} \leq \lambda < 1$ or $\lambda > 1$ respectively, whenever $\alpha \geq 2$. Similar results are obtained for functions related to the Turán expression, and in passing, an integral representation is derived for $\Delta_n^{(\frac{1}{2})}(x)$ ($n = 1, 2, \cdots$). *C. A. Swanson.*

Rot, André. Les fonctions d'Airy. Ann. Télécommun. 12 (1957), 343–346.

This paper summarizes properties of the Airy Functions $\mathrm{Ai}\, x$ and $\mathrm{Bi}\, x$, which are solutions of the equation $y'' = xy$, and includes their relation with Bessel Functions of order $\pm\frac{1}{3}$. [See Miller, The Airy Integral, Cambridge, 1946; MR 8, 353.]

Related functions $\mathrm{Gi}\, x$, $\mathrm{Hi}\, x$, introduced by R. S. Scorer [Quart. J. Mech. Appl. Math. 3 (1950), 107–112; MR 12, 287] are also considered. These are solutions of $y'' - xy = \mp 1/\pi$ with sum $\mathrm{Bi}\, x$. Integrals of these form functions which are included. The account also has a useful bibliography. *J. C. P. Miller* (London).

Kreyszig, Erwin. On the complementary functions of the Fresnel integrals. Canad. J. Math. 9 (1957), 500–510.

The functions considered are $c(z) = \int_z^\infty t^{-\frac{1}{2}} \cos t\, dt$ and $s(z) = \int_z^\infty t^{-\frac{1}{2}} \sin t\, dt$. The author shows the following. (i) The usefulness of the asymptotic expansions of $c(z)$ and $s(z)$ for large positive z can be increased by suitable re-expansion. The method used is essentially that of J. R. Airey [Phil. Mag. (7) 24 (1937), 521–552]. (ii) The zeros of $c(z)$ and $s(z)$ are real and lie in the intervals $n\pi \leq z \leq (n+\frac{1}{2})\pi$, $(n+\frac{1}{2})\pi \leq z \leq (n+1)\pi$, respectively, where $n = 0, 1, \cdots$.

The first 26 zeros of each function are tabulated to 2 decimal places, and asymptotic expansions are given for the large zeros. Also included are brief tables of $c(z)$ and $s(z)$ for real and complex arguments, and sketches of the surfaces of $|c(z)|$, $|s(z)|$ against $z = x + iy$. *F. W. J. Olver.*

See also: Analytic Theory of Numbers: Iseki; Mikolás. **Functions of Complex Variables:** Cowling and Thron. **Numerical Methods:** Barlett, Rice and Good; Fröberg and Wilhelmsson; Longman.

Sequences, Series, Summability

Hsiang, Fu Cheng. An inequality for finite sequences. Math. Scand. **5** (1957), 12–14.

It is known that $\sum_{\mu=0}^{m}\sum_{\nu=0}^{m}|a_\mu b_\nu|/(2\mu+2\nu+1)\leqq K_m(\sum_0^m|a_\mu|^2)^{\frac{1}{2}}(\sum_0^m|b_\nu|^2)^{\frac{1}{2}}$ with $K_m\leqq\pi$ [G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, 1934; p.226]. The author gives a very simple proof that the upper bound on K_m can be improved to $(m+1)\sin[\pi/(m+1)]$. *A. E. Livingston* (Seattle, Wash.).

Makai, E. On the inequality of Mathieu. Publ. Math. Debrecen **5** (1957), 204–205.

Die Ungleichung $\sum_{n=1}^{\infty}2n/(n^2+x^2)^2<1/x^2$ die von E. Mathieu [Traité de physique mathématique, t. VI–VII. partie 2, Gauthier-Villars, Paris, 1890] vermutet, von K, Schröder [Math. Ann. **121** (1949), 247–326; MR **12**, 185] und von O. Emersleben [ibid. **125** (1952), 165–171; MR **14**, 369] teilweise, von L. Berg [Math. Nachr. **7** (1952), 257–259; MR **14**, 731] und von J. G. van der Corput und L. O. Heflinger [Nederl. Akad. Wetensch. Proc. Ser. A. **59** (1956), 15–20; MR **17**, 949] mittels mehr schwierigen analytischen Hilfsmitteln vollständig bewiesen wurde, wird hier so einfach gewonnen, dass in diesem Referat der ganze Beweis wiedergegeben werden kann: Es wird die evidente Ungleichung $[(n-\frac{1}{2})^2+x^2-\frac{1}{4}]^{-1}-[(n+\frac{1}{2})^2+x^2-\frac{1}{4}]^{-1}>2n/(n^2+x^2)^2$ bezüglich n von 1 bis ∞ summiert. Mit derselben Methode wird auch die anderseitige Abschätzung $\sum_{n=1}^{\infty}2n/(n^2+x^2)^2>1/(x^2+\frac{1}{2})$ bewiesen, die für $x^2>5/6$ schlechter, für $x^2<5/6$ dagegen besser ist als die untere Abschätzung $\sum_{n=1}^{\infty}2n/(n^2+x^2)^2>1/x^2-5/(16x^4)$ von O. Emersleben. *J. Aczél* (Debrecen).

Lunc, G. L. On a class of generalized Dirichlet series. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 173–179. (Russian)

The author studies series (*) $\sum_{n=1}^{\infty}a_ne^{-\lambda_n s}$ $(s=\sigma+i\tau)$ with complex exponents satisfying $|\lambda_n|\leqq|\lambda_{n+1}|\to\infty$. Let n_k run through the integers for which $|\arg\lambda_n-\varphi|<\alpha$, put

$$k(\varphi,\alpha)=\limsup|\lambda_{n_k}|^{-1}\log|a_{n_k}|,$$

$$m(\varphi,\alpha)=\limsup|\lambda_{n_k}|^{-1}\log|ka_{n_k}|,$$

and denote by $k(\varphi)$ and $m(\varphi)$ the limits of these expressions as $\alpha\searrow 0$. Using estimates similar to those employed in the study of Dirichlet series with real exponents, the author proves: I. (*) is divergent at $\sigma+i\tau$ if $\sigma\cos\varphi-\tau\sin\varphi<k(\varphi)$ for at least one real φ; II. if $\log n=O(|\lambda_n|)$, then (*) is absolutely convergent at $\sigma+i\tau$ if $\sigma\cos\varphi-\tau\sin\varphi>m(\varphi)$ for all real φ. He also gives sufficient conditions that II yield the maximal open set of absolute convergence.

This paper is related to an earlier one by the author [Mat. Sb. N.S. **10(52)** (1942), 33–50; MR **4**, 218]. Special cases of the results are due to E. Hille [Ann. of Math. (2) **25** (1924), 261–278] and C. Yu. [Acta Math. Sinica **5** (1955), 295–311; MR **19**, 265]. *A. Dvoretzky.*

Basu, S. K. On comparison of the total strength of some Hausdorff methods. Math. Z. **67** (1957), 303–309.

In Fortsetzung früherer Untersuchungen [z.B. Amer. J. Math. **76** (1954), 389–398; Proc. Amer. Math. Soc. **5** (1954), 226–238; MR **15**, 697] betrachtet der Verfasser die zu den beiden Hausdorff-Matrizen mit den Diagonalfolgen $\mu_n=(\beta/(n+\beta))^\alpha$ $(\alpha,\beta>0)$ bzw. $\mu_n=e^{-\beta n^{\frac{1}{2}}}$ $(\beta>0)$ gehörigen Limitierungsverfahren $\Gamma_\beta{}^\alpha$ bzw. J^β und vergleicht dieselben 'total' mit dem Hölder-Verfahren $H^\alpha=\Gamma_1{}^\alpha$ (Vergleich der Verfahren auch bei Limitierbarkeit gegen $+\infty$). Die Verfahren $\Gamma_\beta{}^\alpha$ und H^α sind äquivalent [G. H. Hardy, Divergent series, Oxford, 1949, S. 265; MR **11**, 25]; darüber hinaus wird gezeigt: $\Gamma_\beta{}^\alpha$ ist total stärker als H^α, aber H^α ist nicht total stärker als $\Gamma_\beta{}^\alpha$ $(0<\beta<1;\alpha>0)$; H^α ist total stärker als $\Gamma_\beta{}^\alpha$, aber $\Gamma_\beta{}^\alpha$ ist nicht total stärker als H^α $(\beta>1,\alpha>0)$. Weiter wird der totale Vergleich der Verfahren J^β und H^α für gewisse Kombinationen von α,β durchgeführt. *D. Gaier* (Stuttgart).

Borwein, D. On methods of summability based on power series. Proc. Roy. Soc. Edinburgh. Sect. A. **64** (1957), 342–349.

Ausgehend von einer Potenzreihe $p(x)=\sum_0^\infty p_nx^n$ mit $p_n\geqq 0$, $\sum_{\nu=n}^{\infty}p_\nu>0$ $(n=0,1,\cdots)$ und Konvergenzradius $\rho_p>0$ nennt der Verf. eine Folge $\{s_n\}$ P-summierbar zum Wert s, falls $p_s(x)=(p(x))^{-1}\sum_0^\infty p_ns_nx^n$ für $x\to\rho_p-0$ gegen s strebt. Als Hauptergebnis der Arbeit wird gezeigt, dass für zwei derartige Verfahren die Inklusion $Q\subseteqq P$ gilt, falls (a) $p_n=q_n\int_0^1t^nd\chi(t)$ ist mit $\int_0^1t^nd|\chi(t)|\leqq K\int_0^1t^nd\chi(t)$ $(n=N,N+1,\cdots)$ und (b) P regulär ist und $\rho_p=\rho_q$ ist. Ist $p_n{}^\alpha=\binom{n+\alpha}{n}$ $(\alpha>-1)$, $p_n{}^{-1}=(n+1)^{-1}$, und werden die zugehörigen Verfahren mit A_α bezeichnet (A_0 ist das Abelverfahren), so ergibt sich die Beziehung $A_\lambda\supseteqq A_\mu(\mu>\lambda\geqq-1)$. Ist $p_0{}^\alpha=1$, $p_n{}^\alpha=((\alpha+1)(\alpha+2)\cdots(\alpha+n))^{-1}$ $(\alpha>-1)$, und werden die zugehörigen Verfahren mit B_α bezeichnet (B_0 ist das Borelsche Verfahren), so folgt $B_\mu\supseteqq B_\lambda$ $(\mu>\lambda>-1)$. *A. Peyerimhoff* (Giessen).

Parameswaran, M. R. Some product theorems in summability. Math. Z. **68** (1957), 19–26.

Let S and T be summability schemes for sequences, with T sequence-to-sequence, and let $S\cdot T$ denote the scheme for sequences obtained by applying S to the T transform of these sequences. O. Szasz [Ann. Soc. Polon. Math. **24** (1952), 75–84; MR **15**, 26] showed that $S\not\subset S\cdot T$ (every sequence summable S is also summable $S\cdot T$) when S is Abel, Borel, or regular Hausdorff and T is regular Hausdorff; he also exhibited S and T for which $S\subset S\cdot T$. V. Vučković [Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 53–58; MR **17**, 961] was concerned with the reverse inclusion and showed that $S\cdot T\subset S$ when S is Abel or regular Nörlund and $(Ts)_n=\lambda_0s_{n-1}+\lambda_1s_n$ for complex λ_0 and λ_1, with $\operatorname{Re}\lambda_0>\frac{1}{2}$. The author of the paper under review treats both types of inclusion for certain combinations of S as Abel, Borel, regular Hausdorff, or regular Nörlund, and T the method given by $t_n=\sum_{k=0}^{n}\lambda_ks_{n-k}/\sum\lambda_j$, $\{\lambda_j\}_0{}^\infty$ being a fixed real sequence for which $\sum\lambda_j$ is convergent with sum $\neq 0$. For example, he shows that $S\subset S\cdot T$ when S is Abel or Borel (if, also, $\sum|\lambda_j|<\infty$, then S may be regular Nörlund), and that $S\cdot T\subset S$ when S is regular Nörlund and $\sum_1^\infty|\lambda_j|<|\lambda_0|$. For his proof of this latter inclusion relation (when S is the regular Nörlund method defined by the sequence $\{p_j\}_0{}^\infty$ with partial sums $\{P_n\}_0{}^\infty$), he first shows that if $\sum_1^\infty|\lambda_j|<|\lambda_0|$, then the method U defined by $(Us)_n=\sum_{k=0}^{n}\lambda_{n-k}P_ks_k/P_n$ transforms every null sequence into a null sequence (obvious), and has an inverse with the same property; a consequence is that the sequence s is summable U if and only if it is convergent. *A. E. Livingston* (Seattle, Wash.).

Parameswaran, M. R. Some applications of Banach functional methods to summability. Proc. Indian Acad. Sci. Sect. A. **45** (1957), 377–384.

The author considers the set (c) of convergent sequences, its transform $R_{(c)}(A)$ by a matrix A and the closure $\bar{R}$ of

the latter set. He generalizes results of Banach [Théorie des opérations linéaires, Warszawa, 1932, pp. 90–95] and Hanaï [Tôhoku Math. J. (2) 2 (1950), 64–67; MR 12, 695] and simplifies some proofs. Th. 1.1: If A is convergence-preserving and $\chi(A)=\lim_i \sum_n a_{in}-\sum_n \lim_i a_{in}\neq 0$; if further $y\in(c)$ such that $\sum a_t y_t=0$ whenever (2.3) $\sum |a_t|<\infty$ and $\sum_t a_t a_{tn}=0$ for $n=1, 2, \cdots$; then $y\in \bar{R}$. Th. 2 is a converse of Th. 1.1. From these results the author deduces conditions for matrices A of type M, i.e., matrices where (2.3) is true only if all a_t vanish. Other applications concern inverses of matrices; e.g. Th. 6.1: If A satisfies the conditions of Th. 1.1 and has a left inverse $'A$ transforming every bounded sequence into a bounded sequence, then any bounded A-summable sequence is convergent. A recent paper by A. Wilansky and the reviewer [J. London Math. Soc. 32 (1957), 397–408; MR 19, 646] deals with similar topics.

K. Zeller (Tübingen).

Favard, J. Sur la saturation des procédés de sommation. J. Math. Pures Appl. (9) 36 (1957), 359–372.

The notion of a class of saturation relative to a process of summation is applied in a Banach space framework. [See Favard, Ann. Mat. Pura Appl. (4) 29 (1949), 259–291; MR 11, 669; and Butzer, C. R. Acad. Sci. Paris 243 (1956), 1473–1475; MR 18, 585.] Let E be a separable Banach space. Let $\{x^i\}$ be a closed sequence in E, and $\{f_j\}$ a total sequence of functionals with $f_j(x^i)=\delta_{ij}$. Consider a summation process for elements $x\in E$ given by $x=\sum_{i=1}^{\infty} \gamma_i f_i(x)x^i$. If $\varphi(n)\downarrow 0$ and $|1-\gamma_i|<K\varphi(i)$ for some constant K and $i=1, 2, \cdots$, then there exists a class of saturation relative to the summation process. *P. Civin.*

See also: Measure, Integration: Gagliardo.

Approximations, Orthogonal Functions

Stein, Elias M. Interpolation in polynomial classes and Markoff's inequality. Duke Math. J. 24 (1957), 467–476.

The author proves a number of inequalities of Markoff type relating the size of a polynomial with that of its derivative. Let

$$L(f)=(1-x^2)d^2f/dx^2-[(\alpha-\beta)-(\alpha+\beta+2)x]df/dx,\quad (\alpha,\beta>-1),$$

$$\|f\|_p=[\textstyle\int_{-1}^{1} |f(x)|^p(1-x)^\alpha(1+x)^\beta dx]^{1/p}.$$

Then, if f is a polynomial of degree n

$$(1)\qquad \|L(f)\|_p\leq An^2\|f\|_p \quad (1\leq p\leq\infty),$$

$$(2)\qquad \int_{-1}^{+1} |f'(x)|^p(1-x^2)^{\frac{1}{2}\lambda}dx\leq A_\lambda{}^p n^{2p}\int_{-1}^{+1} |f(x)|^p(1-x^2)^{\frac{1}{2}}dx \quad (\lambda\geq -1,\ p\geq 1).$$

A depends only on α, β, and A_λ only on λ. *P. J. Davis.*

Fernandez Avila, Francisco Javier. Generalization of vector spaces by means of the Stieltjes integral. Rev. Mat. Hisp.-Amer. (4) 17 (1957), 22–37, 150–160, 209–223, 278–290. (Spanish)

This paper is an exposition of certain of the more elementary aspects of the theory of systems of functions orthogonal with respect to a measure on an interval of the real number line. The author's "generalization" consists in his replacing in the statements of certain well-known concepts and theorems the phrase "weight function w" by the phrase "measure α". *A. E. Livingston.*

Morgenthaler, George W. On Walsh-Fourier series. Trans. Amer. Math. Soc. 84 (1957), 472–507.

The Walsh system $\{\psi_n(x)\}$ is the completion of the Rademacher system, and can also be identified with the set of characters of the dyadic group; Walsh-Fourier series are analogous to Fourier trigonometric series in many respects, but there are also striking differences. The dyadic group G consists of sequences of 0's and 1's, group addition $(\dot{+})$ being mod 2 in each component. If $\bar{x}=\{x_n\}\in G$, there is a real number $\lambda(\bar{x})=.x_1x_2\cdots$ (scale of 2); let $\mu(x)$ be the inverse function, with domain (0, 1). Put $\lambda(\mu(y)\dot{+}\mu(z))=y\dot{+}z$. With each real function $g(x)$ on the real numbers mod 1 associate the function $\bar{g}(\bar{x})$ on G such that $\bar{g}(\bar{x})=g(x)$ if $\mu(x)=\bar{x}$, and $\bar{g}(\bar{x})=\lim\sup \bar{g}(\bar{y})$ if $\mu(x)\neq\bar{x}$ for any x, the lim sup being taken over $\bar{y}\to\bar{x}$ through $\bar{y}$ corresponding to dyadic irrationals. Fine [same Trans. 65 (1949), 372–414; MR 11, 352] showed that the only absolutely continuous functions whose Walsh-Fourier coefficients are $o(1/k)$ are constants, in sharp contrast with the trigonometric case. The author shows that "on the average" the classical situation obtains: if b_k are the Walsh-Fourier coefficients of the integral of a periodic function of mean value zero then the arithmetic means of $\{k|b_k|\}$ tend to zero. On the other hand, more smoothness in the function is not reflected in better behavior of the coefficients: constants are the only twice-differentiable functions for which the arithmetic means of $\{k^2|b_k|\}$ tend to zero. In the trigonometric case, if the coefficients of a function of bounded variation are $o(1/k)$, the function is continuous; this fails trivially in the Walsh case, but there is an analogue using continuity in G: if $f(x)$ is nondecreasing and bounded on $0\leq x\leq 1$ and its Walsh-Fourier coefficients of index 2^n are $o(2^{-n})$ then $f^*(x)=f(x+0)$ has the same coefficients as $f(x)$ and its G-extension $\bar{f}(\bar{x})$ is continuous on G. Fine [Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 588–591; MR 17, 31] showed that the Walsh-Fourier series of an integrable function is summable (C,α) almost everywhere for $\alpha>0$. The author shows that the $(C, 1)$ means converge in mean (of order 1) to f. He defines a class Lip α on G in a natural way, and a class Lip $\alpha(W)$ on the reals mod 1 by $|f(x\dot{+}y)-f(x)|<Cy^\alpha$, where $\mu(x)+\mu(y)$ does not end in a sequence of 1's, and establishes the connection between these classes. This lets him prove that if $f\in$ Lip $\alpha(W)$ and $f(x-0)$ exists and is finite at dyadic rationals, then the W-modulus of continuity (defined in the natural way) of the partial sums of its Walsh series is uniformly $O(\delta^\alpha \log(1/\delta))$; the proof is via the corresponding result for the Fourier expansion of $\bar{f}(\bar{x})$ in characters of G. Similarly there is a pair of conditions for functions to be Lipschitzian in terms of $(C, 1)$ means of the expansions. Corresponding to a theorem of Rogosinski on trigonometric Fourier series, the author proves that, for a Walsh series on G with partial sums $\bar{S}_n$, which converges to $\bar{s}(\bar{x})$, if $\lambda(\bar{\alpha}_n)=O(1/n)$ we have $\bar{S}_n(\bar{x}\dot{+}\bar{\alpha}_n)\to\bar{s}(\bar{x})$. The strict analogue of another of Rogosinski's theorems is false, but is replaced by: if $f\in L^p(0, 1)$ $(1<p<\infty)$ and $\alpha_n=O(1/n)$, then

$$S_n(x\dot{+}\alpha_n)-[S_n(x)-f(x)]\psi_{n-1}(\alpha_n)\to f(x)$$

almost everywhere. If $f(x)\in L$ and lim sup $s_n(x)<+\infty$ on a set E of positive measure then almost everywhere on E we have lim inf $S_n(x)>-\infty$ and $f(x)=\frac{1}{2}\{\lim\inf S_n(x)+\lim\sup S_n(x)\}$. The author then investigates the trans-

formation of Walsh-Fourier series by multipliers. Since there are difficulties with Walsh-Fourier-Stieltjes series because of the discontinuities of the $\psi_n(x)$, he uses the class of such series for which the integrator function is continuous as well as of bounded variation. These series have much the same multiplier properties as in the trigonometric case. Many familiar theorems on lacunary trigonometric series are shown to hold for lacunary Walsh series. Also the central limit theorem [Salem and Zygmund, ibid. 33 (1947), 333–338; MR 9, 181] holds for Walsh lacunary series; new methods are required. Finally the author constructs a continuous function of bounded variation whose Walsh-Fourier-Stieltjes coefficients do not tend to zero. *R. P. Boas, Jr.* (Evanston, Ill.).

Fine, N. J. Fourier-Stieltjes series of Walsh functions. Trans. Amer. Math. Soc. **86** (1957), 246-255.

Let $s_n(x)$ and $\sigma_n(x)$ denote the partial sums and the $(C,1)$ sums at the point x of the Walsh series (*) $\sum a_k\psi_k(x)$. Morgenthaler has shown [see the paper reviewed above] that $\int_0^1 |\sigma_n(x)|dx=O(1)$ and $s_n(x)=o(n)$ uniformly in [0, 1] are necessary and sufficient for (*) to be a Riemann-Stieltjes series of a continuous determining function. It is now shown that a necessary and sufficient condition that (*) be a Walsh-Stieltjes series corresponding to a measure m is $\int|\sigma_n(x)|dx=O(1)$ and $s_n(\rho-0)=o(n)$ for each dyadic rational ρ. A similar result is obtained for the Walsh series defined on the dyadic group. *P. Civin.*

Koosis, Paul. A completeness theorem. Portugal. Math. **15** (1956), 111–113 (1957).

Let I be the supporting interval of a measure $m(x)$ of compact support. Let a_k $(k=1, 2, \cdots)$ be the different zeros of the Fourier transform $\int_{-\infty}^{\infty} e^{i\lambda x}dm(x)$, a_k occurring with multiplicity n_k+1. The author shows that the set

$$X=\{x^n e^{ia_kx}|\ n=0, \cdots, n_k;\ k=1, 2, \cdots\}$$

is uniformly complete on every interval J whose length is less than that of I. The author also remarks that his result can be considered as an immediate consequence of Levinson's theorems on the density of the zeros of an entire function of exponential type which is close to bounded on the real axis [Gap and density theorems, Amer. Math. Soc. Colloq. Publ., vol. 26, New York, 1940, ch. III; MR 2, 180]. *J. Korevaar* (Madison, Wis.).

See also: Probability: Karlin and McGregor.

Trigonometric Series and Integrals

Lauwerier, H. A. On certain trigonometrical expansions. Math. Centrum Amsterdam. Afd. Toegepaste Wisk. Rep. **TW 43** (1957), i+20 pp.

Let $f(x)$ be defined over the interval $0<x<\pi$. The author studies the problem of expanding $f(x)$ in trigonometric series of the types

$$(1)\qquad f(x)=\sum_{k=0}^{\infty} a_k(\sin kx+\gamma_k \cos kx),$$

$$(2)\qquad f(x)=\sum_{k=1}^{\infty} b_k(\sin kx+\gamma_k \cos kx),$$

where $\{\gamma_k\}$ is a prescribed sequence of "phases". Two cases are considered, first that of "constant phase" where $\gamma_k=\gamma$ for all k (γ real or complex) and second that of "variable phase" where, for some $\alpha>1$,

$$\gamma_k=\gamma+O(k^{-\alpha})$$

as $k\to\infty$ (γ real). Formulae are derived for the coefficients a_k and b_k in both cases. For the first case the author employs conformal mapping; for the second he reduces his problem to solving a Fredholm equation with bounded kernel.The discussion on Hölder conditions is confusing and it is not clear, to the reviewer, what conditions on $f(x)$ are sufficient to guarantee the existence of representations (1) and (2). *F. W. Gehring.*

Kumari, Sulaxana. On the order of the Cesàro means of Fourier series and its successively derived series. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 199–216.

Let $f(t)$ be an integrable function with period 2π, and let $\varphi(t)=\varphi_0(t)=f(x+t)+f(x-t)-2s$ and $\varphi_\alpha(t)=\alpha t^{-1}\int_0^t (1-u/t)^{\alpha-1}\varphi(u)du$, for $\alpha>0$. By $s_n^\alpha(t)$, we denote the nth Cesàro mean of the Fourier series of $f(t)$, and we suppose that $\alpha\geqq 0$ and $-1<p<\infty$. Then the author proves that: (I) if $\int_t^\pi |\varphi_\alpha(t)|t^{-1}dt=o((\log 1/t)^{p+1})$ as $t\to+0$, then $s_n^\alpha(x)=o((\log n)^{p+1})$; (II) there is an even integrable function $\varphi(t)$ such that $\varphi_\alpha(t)=o((\log 1/t)^p)$ as $t\to 0$, and $s_n^\alpha(0)>\xi_n(\log n)^{p+1}$ for any ξ_n which tends to zero as $n\to\infty$; and (III) there is an even periodic integrable function $\varphi(t)$ such that

$$\int_0^t|\varphi_\alpha(u)|du=O(t(\log 1/t)^p),\ \int_0^t\varphi_\alpha(u)du=o(t(\log 1/t)^p),$$

as $t\to 0$, but $s_n^\alpha(0)\neq o((\log n)^{p+1})$. (I) is a generalization of Sunouchi's theorem [Tôhoku Math. J. 3 (1951), 114–122; MR 13, 228]. *S. Izumi* (Chicago, Ill.).

Izumi, Shin-ichi; and Kinukawa, Masakiti. Fourier series. IX. Strong summability of the derived Fourier series. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1957), 145–165.

Complete proofs are supplied for results announced earlier [Proc. Japan Acad. 31 (1955), 107–110; MR 17, 32] on the H_2 summability of the derived Fourier series. Additional results are presented for the strong summability of the derived conjugate series. *P. Civin.*

Ishiguro, Kazuo. Correction to the paper "Fourier series. XI. Gibbs' phenomenon". Kōdai Math. Sem. Rep. **9** (1957), 191–192.

This is the correction referred to at the end of the review [MR 19, 268] of the author's paper in same Sem. Rep. 8 (1956), 181–188.

Ishiguro, Kazuo. Fourier series. XV. Gibbs' phenomenon. Proc. Japan Acad. **33** (1957), 119–123.

The author adds another criterion of similar kind to his earlier results [Kōdai Math. Sem. Rep. 8 (1956), 181–188; MR 19, 268], to the effect that the (C,r)-means of the Fourier series of a function should show, even at a point of discontinuity of the second kind, a Gibbs phenomenon for $r<r_0$, but not for $r\geqq r_0$, where r_0 is Cramér's constant. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Musielak, Julian. On absolute convergence of Fourier series of some almost periodic functions. Zeszyty Nauk. Uniw. Mickiewicza. Mat.-Chem. **1** (1957), 9–17. (Polish. Russian and English summaries)

The author considers the Fourier series $a_0/2+\sum a_n \cos \lambda(n)x+b_n \sin \lambda(n)x$ and investigates their absolute convergence. He puts $M\{g(x)\}=\lim_T (2T)^{-1}\int_{-T}^{+T} g(x)dx$

and $\omega_2(h)=[\sup_{|\delta|\le h} M\{|f(x+\delta)-f(x)|^2\}]^{\frac12}$; further, $\mu(x)$ is the inverse function to the (monotonic) function $\lambda(x)$. Theorem 1: If (4) $\sum_\nu [\mu(2^\nu\pi)-\mu(2^{\nu-1}\pi)+1]^{1-\gamma/2}\omega_2{}^\gamma(2^{-\nu})<\infty$ for a function $f\in B^2$ and a certain number γ in $0<\gamma<2$ then the series (3) $\sum_n (|a_n|^\gamma+|b_n|^\gamma)$ is convergent. — In the hypothesis of the second theorem, ω_2 is replaced by a more complicated "continuity expression". Simpler theorems are true in case $n^\rho=O(\lambda_n)$ for a certain $\rho>0$. For instance, (4) in Theorem 1 can be replaced by $\sum_n n^{(1-\gamma/2)/\rho-1}\omega_2{}^\gamma(n^{-1})<\infty$ [Th. 3; special cases are due to Szász and Bernstein; see Zygmund, Trigonometrical series, Warszawa-Lwów, 1935, p. 177; and Stečkin, Mat. Sb. N.S. **29(71)** (1951), 225–232; MR **13**, 229; **15**, 28]. Finally, two theorems deal with lacunary series where $\lambda(n+1)/\lambda(n)>q>1$. *K. Zeller* (Tübingen).

Følner, Erling. Besicovitch almost periodic functions in arbitrary groups. Math. Scand. 5 (1957), 47–53.

Leading is the following lemma. On any infinite group G there are denumerably many disjoint symmetric ($E=E^{-1}$) subsets $E_1, E_2, \cdots$, such that, to finitely many elements $a_1, \cdots, a_N$ of G, and any $h=1, 2, \cdots$, there is an $x\in G$ for which all products $a_n x a_m$ are in E_h. With this, the author can construct a generalization of Besicovitch almost periodic functions B^p for all $p\geqq 1$ on G. However, the author emphasizes that his universal construction, if applied to $-\infty<x<\infty$ (with topology?), does not give Besicovitch's own objects. *S. Bochner.*

See also: Measure, Integration: Timan.

Integral Transforms

de Branges, Louis. Local operators on Fourier transforms. Duke Math. J. 25 (1958), 143–153.

Let K be a measurable function on $(-\infty, \infty)$. The author defines an operator $K(H)$ whose domain is a subset of the collection of Fourier transforms of functions in $L_1(-\infty, \infty)$, as follows. If $f(x)=\int_{-\infty}^{\infty} e^{ixt}F(t)dt$, with $F\in L_1(-\infty, \infty)$, then f is in the domain of $K(H)$ if $KF\in L_1(-\infty, \infty)$; and $K(H)f(x)=\int_{-\infty}^{\infty} e^{ixt}K(t)F(t)dt$.

The author calls $K(H)$ "local" if whenever f_1 and f_2, in the domain of $K(H)$, take on the same values in a neighbourhood of x_0, then $K(H)f_1(x)$ and $K(H)f_2(x)$ take on the same values at x_0. The author shows that a sufficient condition that $K(H)$ be local is that $K(x)$ agree a.e. with the values taken on on the real axis by an entire function of minimal exponential type; he shows that this condition is necessary if $K(H)$ has in its domain a function which vanishes throughout an interval without vanishing identically.

The author also shows that if K is a measurable function, with $|K|$ steadily increasing, such that
$$\int_{-\infty}^{\infty} \frac{\log^+|K(t)|}{1+t^2}\,dt=\infty,$$
and if μ is a measure of finite total variation such that $\int_{-\infty}^{\infty}|K(t)\mu(dt)|<\infty$, then $\int_{-\infty}^{\infty} e^{ixt}\mu(dt)$ cannot vanish in an interval without vanishing identically. *P. G. Rooney.*

Ragab, F. M. The inverse Laplace transform of an exponential function. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-107** (1957), 16 pp.

The inverse of the Laplace transform
$$p^{\alpha-1}\exp(-a^{1/m}p^{1/m})$$
is found in terms of MacRobert's E-function (which is closely related to the generalized hypergeometric function), for $\mathrm{R}(\alpha)$, $\mathrm{R}(a)>0$ and $m=2, 3, 4, \cdots$. The asymptotic expansion of the inverse, valid near the origin, is also found. Both results reduce to the known one for $m=2$. It is expected that these results will be useful in electromagnetic diffraction problems. *T. E. Hull.*

Fox, Charles. A composition theorem for general unitary transforms. Proc. Amer. Math. Soc. 8 (1957), 880–883.

The following is a well-known theorem of S. Bochner [Ann. of Math. (2) **35** (1934), 111–115]. To each unitary transformation T in $L_2(0, \infty)$, such that $Tf=g$, there correspond two functions $k(a, x)$, $l(a, x)$ belonging to the class $L_2(0, \infty)$ for each a, $0<a<\infty$, such that
$$\int_0^a g(y)dy=\int_0^\infty \overline{k(a, x)}f(x)dx;\ \int_0^b f(x)dx=\int_0^\infty \overline{l(b, y)}g(y)dy;$$
$$\int_0^\infty k(a, x)\overline{k(b, x)}dx=\min(a, b);\ \int_0^\infty l(a, y)\overline{l(b, y)}dy=\min(a, b);$$
$$\int_0^b \overline{k(a, x)}dx=\int_0^a l(b, x)dx.$$
A pair of 'kernels' such as $k(a, x)$ and $l(a, x)$ may be called "unitary". The author proves the simple result that if $p(a, x)$, $q(a, x)$ are another such pair, then the functions $m(a, x)$, $n(a, x)$ given by
$$m(a, x)=\frac{\partial}{\partial x}\int_0^\infty \overline{k(x, y)}p(a, y)dy,$$
$$n(a, x)=\frac{\partial}{\partial x}\int_0^\infty k(a, y)\overline{p(x, y)}dy$$
are also such a pair. *K. Chandrasekharan.*

Ordinary Differential Equations

Vinograd, R. È.; and Grobman, D. M. On problems of Frommer differentiation. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 5(77), 191–195. (Russian)

The question of the behavior of the trajectories of
$$\frac{dy}{dx}=\frac{P_n(x, y)+p(x, y)}{Q_n(x, y)+q(x, y)}$$
(P_n, Q_n forms of degree n; p, q terms of degree $<n$) has been dealt with by Frommer [Math. Ann. **99** (1928), 222–272] by passing to polar coordinates r, φ:
$$r\frac{d\varphi}{dr}=\frac{F(\varphi)+f(r, \varphi)}{G(\varphi)+g(r, \varphi)}.$$
Let φ_0 be a root of $F(\varphi)$. We may suppose that $\varphi_0=0$. Let $G(0)=b_0\neq 0$. Then one may take
$$F(\varphi)=a_0\varphi^k+a_1\varphi^{k+1}+\cdots\ (k\geqq 1),$$
$$G(\varphi)=-1+b_1\varphi+\cdots.$$

It is known [Frommer, loc. cit.; Nemitzkiĭ and Stepanov, Qualitative theory of differential equations, 2nd ed., Gostehizdat, Moscow-Leningrad, 1949] that: (a) for k odd and $a_0>0$ there is at least one trajectory tending to the origin with the tangent $\varphi=0$; (b) for k even there is an infinity of or no trajectories of the type in question. The author proves the theorem: If $f(r, \varphi)=Ar+\Phi(r, \varphi)$, $g(0, \varphi)=\Phi(0, 0)=0$ and $r\Phi$, rg satisfy in $r^2+\varphi^2\leqq\alpha^2$ a Lipschitz condition with constant $\to 0$ with α, then under (a) there is only one trajectory and under (b) there is an infinity. *S. Lefschetz* (Mexico, D.F.).

Olech, C. On surfaces filled up by asymptotic integrals of a system of ordinary differential equations. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 935–941, LXXIX. (Russian summary)

For the linear system (1) $y'=Ay$, where y is an n-vector and A is a constant $n\times n$ matrix, let μ be any eigenvalue of A. All solutions of (1) are vectors with components of the form $P(t)e^{\lambda t}$, with $P(t)$ a polynomial and λ an eigenvalue of A. Let $\beta-1$ be the highest degree of such polynomials for those λ with $\operatorname{Re}\lambda=\operatorname{Re}\mu$. Let E^I, E^{II}, E^{III} be those subspaces of E^n which are filled by solutions of (1) with $\operatorname{Re}\lambda<\operatorname{Re}\mu$, $\operatorname{Re}\lambda=\operatorname{Re}\mu$, $\operatorname{Re}\lambda>\operatorname{Re}\mu$ respectively. E^{II} is further decomposed into a direct sum of β subspaces, $E^1\dot{+}E^2\dot{+}\cdots\dot{+}E^\beta$, such that $G^i=E^1\times\cdots\dot{+}E^i$ is the subspace filled by solutions of (1), with $\operatorname{Re}\lambda=\operatorname{Re}\mu$, and polynomial factors of degree less than i. For any vector function $x(t)$, write $x(t)=x^I(t)+x^1(t)+x^2(t)+\cdots+x^\beta(t)+x^{III}(t)$, with $x^i(t)\in E^i$ $(i=I, 1, \cdots, \beta, III)$. If $\|x\|$ denotes euclidean length, let

$$\chi_t(x)=\|x^I(t)\|+\|x^{III}(t)\|+\textstyle\sum_{i=1}^{\beta} t^{i-\beta}\,\|x^i(t)\|.$$

The author considers a nonlinear system of the form (2) $x'=Ax+G(x,t)$. If s is any integer $1\leqq s\leqq\beta$, and $y_0(t)$ is that solution of (1) with $y_0(0)=\xi_0\in E^s$, it is shown that with appropriate conditions on $G(x,t)$ the surface generated for $t>0$ by those solutions of (2) such that $\chi_t(x(t)-y_0(t))=o(t^{s-\beta}\exp(\operatorname{Re}\,\mu t))$ is homeomorphic with the hyperplane $E^I\dot{+}E^1\dot{+}\cdots\dot{+}E^{s-1}\times[0,+\infty]$. *W. S. Loud.*

Lidskiĭ, V. B. Conditions for complete continuity of the resolvent of a differential operator. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 28–31. (Russian)

Let $p(x)=q(x)+ir(x)$, where q and r are real functions, bounded and integrable over every finite interval, and let the operator L on $L^2(-\infty,\infty)$ be defined by $Ly=-y''+p(x)y$ with domain D consisting of those elements for which this expression is in $L^2(-\infty,\infty)$. Then it is proved that, if $\lim_{|x|\to\infty}q(x)=\infty$, then the resolvent operator of L is completely continuous wherever it exists. The conclusion also holds if $\lim_{|x|\to\infty}r(x)=+\infty$ or $-\infty$, subject, however, to the reservation (when $q(x)$ is not bounded below) that, if $-y''+p(x)y=\lambda y$ has all its solutions in $L^2(-\infty,\infty)$, the domain of L must be taken to be a subset of D consisting of functions which obey prescribed boundary conditions at infinity. Finally, it is shown that if $q(x)$ is bounded below, and $r(x)$ is bounded either above or below, then it is a necessary and sufficient condition for the complete continuity of the resolvent of L that, for any sequence of nonintersecting intervals D_n, all of the same length, $\int_{D_n}(q(x)+|r(x)|)dx\to\infty$ as the interval tends to infinity. *J. L. B. Cooper* (Cardiff).

Delsarte, J.; et Lions, J. L. Transmutations d'opérateurs différentiels dans le domaine complexe. Comment. Math. Helv. **32** (1957), 113–128.

This paper contains the details of the results announced in a previous note [C. R. Acad. Sci. Paris **244** (1957), 832–834; MR **19**, 273]. H is the space of entire functions, and the main theorem is that for any differential operator of the form $A=D^m+a_1(x)D^{m-1}+\cdots+a_m(x)$, where $D=d/dx$, there is a bicontinuous automorphism X of H such that $XA=D^mX$. Let G be the group of all automorphisms of H, and let G_m be those that commute with D^m. Then, these can be explicitly given. More generally, one may consider G_A, the group of automorphisms Y that commute with A; for a given function $f\in H$, let $T_A(f)$ be the subspace of H which is spanned by the functions Yf for $Y\in G_A$. One says that f is mean-periodic with respect to A if $T_A(f)$ is not dense in H. Schwartz mean-periodic is the special case $A=D$; the authors show that this is the same as mean-periodic D^m, and by the transmutation theorem reduce the general case to this. Application is also made to partial differential operators, of the form of a polynomial in d/dx and d/dy, whose coefficients are functions of the form $a(x)b(y)$; transmutation reduces the existence problem to that for an operator with constant coefficients, to which the Malgrange theorems [Ann. Inst. Fourier, Grenoble **6** (1955–1956), 271–355; MR **19**, 280] can be applied. *R. C. Buck* (Madison, Wis.).

Voznyuk, L. L. Investigations sur la stabilité des solutions périodiques des équations du haut ordre. Ukrain. Mat. Ž. **9** (1957), 235-251. (Russian. French summary)

The equation under discussion is the symbolic one

$$R(p)z=\varepsilon\Phi(z,\varepsilon),\quad p=d/dt, \tag{1}$$

where R is a polynomial (the author takes an entire function but a polynomial is "safer"), ε is positive and small, and Φ is analytic in ε for $\varepsilon<\varepsilon_0$ and has sufficiently many partials in z. Let $z(\omega t)$ be a solution of period $2\pi/\omega$. Its stability is decided by reference to that of the variation equation

$$R(p)x=\varepsilon\Phi'(z(\omega t),\varepsilon)x=\varepsilon F(\omega t,\varepsilon)x. \tag{2}$$

A solution $x=v(\omega t,\varepsilon)\exp qt$ is tried giving

$$R(q+\omega d/d\varphi)v(\varphi,\varepsilon)=\varepsilon F(\varphi,\varepsilon)v(\varphi,\varepsilon),\quad \omega t=\varphi. \tag{3}$$

It is proved that under certain very general conditions (3) has a solution of the form

$$v(\varphi,\varepsilon)=ae^{i\varphi}+a^*e^{-i\varphi}+\varepsilon u_1(\varphi)+\varepsilon^2u_2(\varphi)+\cdots,$$

where the u_j are periodic with period 2π in φ, and no first harmonics in their Fourier series. Complicated computations (forceful use of Fourier expansions) yield an equation in q, and (1) is stable if all the roots of this equation have negative real parts.

A similar treatment is applied to

$$R(p)z=\varepsilon\Phi(z,S(p)z,\varepsilon)$$

(S and Φ polynomials).

The paper is in the spirit of the methods of Krylov-Bogoliubov [see Bogoliubov and Mitropolskiĭ, Asymptotic methods in the theory of non-linear oscillations, Gostehizdat, Moscow, 1955; MR **17**, 368; also Voznyuk, Dopovidi Akad. Nauk Ukrain. RSR **1957**, 13–17; MR **19**, 142]. *S. Lefschetz* (Mexico, D.F.).

Lykova, O. B. On the behaviour of solutions of a system of differential equations in the vicinity of a static solution. Ukrain. Mat. Ž. **9** (1957), 281–295. (Russian. English summary)

Stability of solutions of the system (1) $dx/dt=X(x)+\varepsilon X^*(t,x,\varepsilon)$ is investigated — ε is a small parameter; x, X, and X^* are vectors in E^n. It is assumed that (a) the autonomous system $dx/dt=X(x)$ has an isolated static solution corresponding to the equilibrium point $x=0$, $X(0)=0$ $[X'(0)\neq 0]$; (b) the function $X+\varepsilon X^*$ and all of its partial derivatives with respect to x and ε are bounded and uniformly continuous with respect to x and ε in the region $-\infty<t<\infty$, $0<\varepsilon<\varepsilon_0$, $x\in U$, where U is a neighborhood of $x=0$; (c) X^* is periodic in t with period 2π; and (d) the characteristic equation $|I_nz-X'(0)|=0$ has two pure imaginary conjugate roots and $n-2$ other roots with negative real parts. It is shown that (1) has a local integral

manifold of solutions, depending on two parameters, which is strongly stable. If $X=Px$, where P is a constant $n\times n$ matrix, then the system has the same property independent of the range of initial values considered.

N. D. Kazarinoff (Ann Arbor, Mich.).

Mitropol'skiĭ, Yu. A. On certain differential equations encountered in relaxational oscillation theory. Ukrain. Mat. Ž. 9 (1957), 296–309. (Russian. English summary)

The system under discussion is the n-vector one

(1) $$\dot{x}=X(x)+\varepsilon X^*(\nu t, x, \varepsilon),$$

where X is periodic with period 2π in νt, and X, X^* are both real and indefinitely differentiable in x and ε. Let

(2) $$\dot{x}=X(x)$$

have the stable periodic solution

$$x^0(\omega t+\varphi)=x^0(\psi),\ \psi=\omega t+\varphi,$$

(period 2π in ωt), with the arbitrary constant φ. The variation equation

(3) $$d(\delta x)/dt=X_x'(x^0)\delta x$$

has then one characteristic root zero and the $(n-1)$ remaining with negative real parts. The system (1) is first transformed to

(4) $$\dot{\psi}=\omega+P(\nu t, \psi, h, \varepsilon),\ \dot{h}=Hh+R(\nu t, \psi, h, \varepsilon),$$

where h is an $(n-1)$-vector, R is real and: (a) P and R are defined for all t and all angles ψ and for $\|h\|<\delta$, $0<\varepsilon<\varepsilon_0$ (region Δ); (b) $P(\nu t, \psi, 0, \varepsilon)$ and $R(\nu t, \psi, 0, \varepsilon)$ are of order ε; (c) the characteristic roots of H have negative real parts. A special one-parametric family of solutions of the second equation (4) is found as power series in ε, then substituted in the first equation (4), thus yielding its solution a one-parameter family of solutions in Δ. It is then shown that all solutions of (1) tend to a solution of the family.

S. Lefschetz (Mexico, D.F.).

Gel'fand, I. M.; and Lidskiĭ, V. B. On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients. Amer. Math. Soc. Transl. (2) 8 (1958), 143–181.

Translated from Uspehi Mat. Nauk (N.S.) 10 (1955), no. 1(63), 3–40 [MR 17, 482].

Seifert, George. On stability in the large for periodic solution of differential systems. Ann. of Math. (2) 67 (1958), 83–89.

The author considers the equation (1) $\dot{x}=f(x,t)$, where x is a real two-dimensional vector and f is periodic in t. The bounded, simply-connected region R in the (x_1, x_2) plane is called a relative bound for (1) if it is the closure of a domain, and if, for arbitrary t_0, every solution $x(t)$ such that $x(t_0)\in R$ remains in R for $t>t_0$. The existence of a periodic solution of (1) is implied by the existence of a relative bound, and the author gives additional conditions (too lengthy to be presented here) which insure its uniqueness. The method is the consideration of the boundary of R, defined parametrically by $x=z(u)$, $0\leqq u<2\pi$. Let $x(t,u)$ be the solution of (1) such that $x(t_0,u)=z(u)$ and let $\Gamma_{t_0}(t)$ denote the simple closed curve defined for $t\geqq t_0$ by the points $x(t,u)$, $0\leqq u<2\pi$. Under the conditions given, it is shown that the length of $\Gamma_{t_0}(t)$ tends to zero as $t\to\infty$, thus implying the uniqueness of the periodic solution. The result is applied to a system equivalent to the equation $\ddot{x}+f(x)\dot{x}=g(x)+p(t)$. *C. E. Langenhop* (Ames, Iowa).

Strelkov, S. P. Application of Galerkin's method to problems of self-oscillations. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. 12 (1957), no. 3, 51–55. (Russian)

The author discusses a formulation of Galerkin's approximation method as applied to the problem of finding approximations to periodic solutions of self-oscillatory systems. The method consists of assuming a solution of the form $\phi(t, a_1, a_2, \cdots)$, where the form of ϕ is prescribed and the parameters $a_1, a_2, \cdots$ are to be determined. A variational principle is used to obtain equations for the parameters $a_1, a_2, \cdots$. The accuracy of the approximation depends upon the form selected for ϕ. The author shows that the results obtained are the same as those obtained by using methods of Andronov and Haĭkin [Theory of oscillations, ONTI, Moscow-Leningrad, 1937] and van der Pol [Phil. Mag. (7) 3 (1927), 65–80]. A measure of error is defined, and estimates for the error are derived. The method is illustrated for the equation $x''+\omega_0^2x=\mu(\alpha-\gamma x^2)x'$. *W. S. Loud.*

Kondratiev, V. A. Sufficient conditions for non-oscillatory or oscillatory nature of solutions of 2nd order equation $y''+p(x)y=0$. Dokl. Akad. Nauk SSSR (N.S.) 113 (1957), 742–745. (Russian)

It is known that the solutions of the equation $y''+p(x)y=0$ are non-oscillatory for $x\geqq x_0$ if there exists a continuously differentiable function θ such that $\theta'+\theta^2+p\leqq 0$, $x\geqq x_0$. Using this the author gives quite a few sufficient conditions for the solutions to be non-oscillatory or oscillatory. The proofs all center about an appropriate choice of θ. *E. A. Coddington* (Princeton, N.J.).

Nikolenko, L. D. Some criteria for non-oscillation of a fourth order differential equation. Dokl. Akad. Nauk SSSR (N.S.) 114 (1957), 483–485. (Russian)

The equation (1) $y^{(IV)}+(ay')'+by=0$ is considered on $x_0\leqq x<\infty$, where a and b are real-valued continuous functions. It is called non-oscillatory if, for some sufficiently large x_0, no non-trivial solution has more than one double zero for $x\geqq x_0$. Using a result analogous to the Sturm comparison theorem, and knowledge of certain special equations, the author indicates several criteria for non-oscillatory equations. Let $\omega(\alpha)=(9/4)\alpha-(9/16)$ for $\alpha\leqq 5/2$, and $\omega(\alpha)=(1/4)(\alpha+2)^2$ for $\alpha\geqq 5/2$. Theorem 1: If there exists an α such that $a(x)\leqq\alpha/x^2$, $b(x)\geqq\omega(\alpha)/x^4$ for all sufficiently large x, then (1) is non-oscillatory. Theorem 2: If for some finite $\alpha\neq 5/2$,

$$\int^\infty x\ln x\max[a(x)-\alpha/x^2, 0]dx<\infty,$$

$$\int^\infty x^3\ln x|\min[b(x)-\omega(\alpha)/x^4, 0]|dx<\infty,$$

then (1) is non-oscillatory. The same conclusion is valid for $\alpha=5/2$ provided $\ln x$ is replaced by $\ln^3 x$ and $\omega(\alpha)=81/16$ in these two integrals. *E. A. Coddington.*

Chan, Chan Khun. The existence and uniqueness of solutions of boundary problems for non-linear ordinary differential equations. Dokl. Akad. Nauk SSSR (N.S.) 113 (1957), 1227–1230. (Russian)

The author states existence and uniqueness results for boundary value problems of the type:

(1) $y^{(n)}=\varphi(x, y, y', \cdots, y^{(n-1)})$, $y^{(k)}(a)=y_0^{(k)}$ $(k=0, 1, \cdots, n-2)$, $y^{(i)}(b)=y_1^{(i)}$, where i is one of the numbers $0, 1, \cdots, n-2$; (2) $u''=F(t, u, u')$, $u(a)=\alpha$, $u(b)=\beta$, where u is an n-dimensional vector; (3) $y''=\varphi(x, y, y')+\alpha y$,

α positive constant, $y(0)=y(1)$, $y'(0)=y'(1)$; (4) $w''=f(x,w,w')$, $w(0)=w(1)=0$.

It is stated that proofs depend on general implicit function theorems, continuation methods, and the Schauder-Leray result. An application is indicated to the problem

$$u_t=u_{xx}-f(x, u, u_x),\ u(0, t)=u(1, t)=u(x, 0)=0.$$

E. A. Coddington (Princeton, N.J.).

Krein, M. G. On a continual analogue of a Christoffel formula from the theory of orthogonal polynomials. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 970–973. (Russian)

For a given Sturm-Liouville system, M. M. Crum [Quart. J. Math. Oxford Ser. (2) **6** (1955), 121–127; MR **17**, 266] showed how to express in determinantal form the "*n*th associated system" and its solutions. The author develops this work, defining an associated system for any polynomial which is non-negative on the spectrum of the original system, and allowing the basic interval to be semi-infinite. The formulae for the solutions, and that connecting the spectral functions of the two systems, are analogous to those in orthogonal polynomial theory which arise when the weight-function is multiplied by a non-negative polynomial. The rest of the paper deals with inverse Sturm-Liouville theory, namely (i) the determination of potentials of the form $p(p-1)/r^2+q(r)$ from the spectral function $\tau(\lambda)$, (ii) a corrected version of earlier conditions on the "transition function"

$$\Pi(t)=\int_{-\infty}^{\infty}(1-\cos t\sqrt{\lambda})\lambda^{-1}d\tau(\lambda),$$

and (iii) a condition on $\tau(\lambda)$ that $\int_{-\infty}^{1} d\tau(\mu)(\mu^2+1)^{-\frac{1}{2}p}$ should be a spectral function of a suitable system, it then following that $\tau(\lambda)$ is a spectral function of an associated system. *F. V. Atkinson* (Canberra).

Sargsyan, I. S. On differentiation of an expansion in eigenfunctions of a Sturm-Liouville operator. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 263–282. (Russian)

This paper contains the detailed proofs of results announced by the author in an earlier note [Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 821–824; MR **17**, 853].

F. Smithies (Cambridge, England).

Kestens, Jean. Le problème aux valeurs propres normal et bornes supérieures et inférieures par la méthode des itérations. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **29** (1956), no. 4, 102 pp.

Consider, on the finite x interval $[a, b]$, the eigenvalue problem consisting of the ordinary differential equation

$$\text{(I)}\qquad M(y)=\lambda N(y),$$

where $M(y)$ and $N(y)$ are linear self-adjoint differential operators of orders $2m$ and $2n$ respectively ($m>n\geq 0$), and the $2m$ linearly independent boundary conditions

$$\text{(II)}\qquad U_\nu(y)=\sum_{i=0}^{2m-1}[\alpha_{\nu i}y^{(i)}(a)+\beta_{\nu i}y^{(i)}(b)]=0\ (\nu=1, \cdots, 2m)$$

with real coefficients $\alpha_{\nu i}$ and $\beta_{\nu i}$. It is supposed that the eigenvalue problem is self-adjoint; that is, that whenever the functions u and v of class $C^{2m}([a, b])$ satisfy the boundary conditions (II) one has $\int_a^b [vM(u)-uM(v)]dx=0$ and $\int_a^b [vN(u)-uN(v)]dx=0$. E. Kamke [Math. Z. **45** (1939), 788–790; **46** (1940), 231–250, 251–286; MR **1**, 235; **2**, 52] has studied a subclass of the class of eigenvalue problems just mentioned. This subclass "of M definite problems" is characterized by the additional restriction that, whenever $u\in C^{2m}([a, b])$ satisfies the boundary conditions (II), one has $\int_a^b uM(u)dx\geq 0$; and further, for all $u\not\equiv 0$ for which $\int_a^b uM(u)dx=0$, one always has either $\int_a^b uN(u)dx>0$ or $\int_a^b uN(u)dx<0$. The author is concerned with the subclass of "N definite problems", which seems to have been overlooked in the previous literature. This subclass of N definite problems is characterized by any one of the three following additional conditions: (a) $\int_a^b uN(u)dx>0$ for $u\not\equiv 0$; (b) $\lambda=0$ is not an eigenvalue, and $\int_a^b uN(u)dx\geq 0$; and (c) $N(y)=g(x)y$, with $g(x)\geq 0$ and $\not\equiv 0$ (or $g(x)\leq 0$ and $\not\equiv 0$). Among other things, the author extends to the N definite class certain of Kamke's results for the M definite class, and establishes upper and lower bounds (by the method of iterations) for the eigenvalues, generalizing results of Kryloff, Bogoliubov and D. H. Weinstein, and of Temple and Collatz [for exact references see L. Collatz, Eigenwertprobleme und ihre numerische Behandlung, Akademische Verlagsgesellschaft, Leipzig, 1945; MR **8**, 514]. Several numerical examples illustrate the typical properties of eigenvalue problems of the M definite and N definite classes. Over and above its original contributions, this memoir constitutes an excellent introduction to the subject. *J. B. Diaz.*

See also: Measure, Integration: Beesack. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Glazman; Nelson; Sahnovič. **Probability:** Karlin and McGregor.

Partial Differential Equations

Lopatinsky, Y. B. On certain problems of the theory of partial differential equations. Ukrain. Mat. Ž. **9** (1957), 389–393. (Russian. English summary)

A brief survey, without formulas, of three topics: approximate solutions, e.g., by finite-difference or probability methods; effective solutions, e.g., "fundamental" solutions for certain types; general theory, e.g., relations among special solutions with given singularities, uniqueness, etc.

Cordes, Heinz Otto. Über die erste Randwertaufgabe bei quasilinearen Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen. Math. Ann. **131** (1956), 278–312.

Consider in a "smoothly" bounded domain $\mathfrak{D}$ (with closure $\bar{\mathfrak{D}}$) in n space with coordinates $x=(x_1, \cdots, x_n)$, the quasilinear elliptic differential equation

$$\text{(1)}\qquad \sum_{i,j=1}^{n} a_{ij}(x, u, u_k)u_{ij}=0$$

where $u_k=\partial u/\partial x_k$, $u_{ij}=\partial^2 u/\partial x_i\partial x_j$, satisfying for all x in $\bar{\mathfrak{D}}$, and all values of the arguments u, u_k, $k=1, \cdots, n$,

$$\text{(2)}\qquad m\sum \xi_i^2\leq \sum a_{ij}\xi_i\xi_j\leq M\sum \xi_i^2 \text{ for real } \xi_1, \cdots, \xi_n$$

with m and M positive constants. One seeks a solution u with given smooth boundary values ϕ. For the case $n=2$ it is known (under appropriate smoothness assumptions on the a_{ij}) that there exists a solution of this Dirichlet problem. This may be proved with the aid of the Schauder fixed point theorem and the Schauder theory for linear elliptic equations. The main point in the proof is a certain a priori estimate for a solution of a linear equa-

tion (1) (a_{ij} independent of u, u_k) satisfying (2). This asserts

(3) $$|u|_{1+\alpha}\equiv \max|u|+\max|u_k|+\sup\frac{|u_k(x)-u_k(y)|}{|x-y|^\alpha}\le K,$$

for a constant K depending only on m, M, the domain $\mathscr{D}$ and the boundary values ϕ. Here $\alpha<1$ is a positive constant depending only on m, M. [See (a) C. B. Morrey, Trans. Amer. Math. Soc. **43** (1938), 126–166; (b) L. Nirenberg, Comm. Pure Appl. Math. **6** (1953), 103–156, 395; MR **16**, 367.]

It is an open question whether this existence theorem, in particular whether the crucial estimate (3), holds in higher dimensions. It is known that (3) can be established under the additional assumption that there is a constant $d>0$ such that in the intersection S of $\bar{\mathscr{D}}$ with every closed sphere of radius d the a_{ij} have small oscillation for all values of the arguments. To be more precise, using results of L. Nirenberg [(c) Contributions to the theory of partial differential equations, Princeton, 1954, pp. 95–100; MR **16**, 592] (3) can be established if in every such intersection S we have (after a possible linear transformation of coordinates, and multiplication of the equation by a factor)

(4) $$\sum_{i,j}(a_{ij}-\delta_{ij})^2<\frac{1}{n-1},$$

or, if we do not multiply by the factor,

(5) $$\frac{\sum a_{ij}^2}{(\sum a_{ii})^2}<\frac{1}{n}+\frac{1}{n^2(n-1)}.$$

(In the same Contributions, pp. 101–159 [MR **16**, 827] Morrey derives estimates analogous to (3) assuming also small oscillation of the coefficients.)

In this paper the author establishes (3), and an existence theorem for the Dirichlet problem, under the less restrictive hypothesis (condition K_ε') that in each such S one assumes, in place of (4), or (5),

(4′) $$\sum(a_{ij}-\delta_{ij})^2<\frac{n^2+n}{2n^2-2n-1},$$

or

(5′) $$\frac{\sum a_{ij}^2}{(\sum a_{ii})^2}<\frac{1}{n}+\frac{n+1}{n(2n^2-2n-1)}.$$

Note that for $n\to\infty$ the right side of (4) tends to zero while that of (4′) tends to $\frac{1}{2}$. The methods used are extensions of those of (a), (b), (c), the desired Hölder condition for the first derivatives, in (3), being derived from an integral estimate: for some $\delta<d$ and every $x^0\in\mathscr{D}$

(6) $$\int\frac{|u(x)-u(x^0)-\sum(x_i-x_i^0)u_i(x^0)|^2}{|x-x^0|^{n+2+2\alpha}}\,dx\le\text{constant},$$

integration being over the intersection of $\mathscr{D}$ with $|x-x^0|<\delta$. This is a variant of an integral estimate used in (a) and, in modified form, in (b) and (c).

The derivation of (6) is rather lengthy and tricky. It is based on certain integral identities involving the Laplacian of u, and uses ideas of E. Heinz [Nachr. Akad. Wiss. Göttingen. IIa. **1955**, 1–12; MR **17**, 626].

L. Nirenberg (New York, N.Y.).

★ **Dressel, F. G.; and Gergen, J. J. The extension of the Riemann mapping theorem to elliptic equations.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 183–195. University of Maryland Book Store, College Park, Md., 1956.

This paper presents a survey of known results concerning homeomorphic mappings of the interior of a simple closed Jordan curve in the (x, y) plane onto the interior of another by solutions u, v of a pair of first order elliptic partial differential equations generalizing the Cauchy-Riemann equations. The mappings are normalized by prescribing the images of three boundary points. The authors describe the work in turn of M. A. Lavrent′ev, Z. Schapiro, C. B. Morrey, L. Bers and L. Nirenberg and their own for linear and nonlinear equations. (References can be found in the paper.)

Lavrent′ev has treated very general nonlinear equations, the conditions on which cannot be described here, and we confine the discussion to quasilinear equations

(1) $$au_x+bu_y=v_y,\quad cu_x+du_y=-v_x,\quad 4ad-(b+c)^2>0,$$

where the coefficients are functions of (x, y), (u, v); if the coefficients are independent of (u, v) the system is linear. Under general conditions Morrey treated linear Beltrami equations ($b=c$), and Bers, Nirenberg, as an application of certain results, using extensions of Morrey's methods, proved a (slightly more refined) version of Schapiro's theorem for quasilinear systems (1). They set up the mapping problem as a boundary value problem for the system (1) in distinction to Lavrent′ev and Shapiro whose work is more geometric.

The authors have treated linear systems (1) by interesting variational methods, and have also proved uniqueness theorems for the mappings.

After this paper the authors have published a paper, together with R. M. McLeod [Duke Math. J. **24** (1957), 173–181; MR **19**, 281] containing a rather general uniqueness theorem for mappings; this is based on an elegant idea (see § 5). This idea has then been further exploited by B. V. Boyarskiĭ in a recent paper [Mat. Sb. N.S. **43(85)** (1957), 451–503] based in part on the ideas of Bers, Nirenberg, although some different technical devices are employed. *L. Nirenberg.*

Nirenberg, Louis. Estimates and existence of solutions of elliptic equations. Comm. Pure Appl. Math. **9** (1956), 509–529.

This paper is a survey of a-priori estimates for solutions of linear elliptic partial differential equations and of the Dirichlet problem for such equations. It contains a comprehensive summary of the results in this direction which had been obtained up to the time of its presentation at the Conference on Partial Differential Equations in Berkeley, California, in June, 1955, and is closely related to the reviewer's paper in the same Comm. **9** (1956), 351–361 [MR **19**, 862]. The first part of the paper discusses the classical case of a single equation of second order, while in the second part extensions to higher order equations and systems of equations are presented. Estimates of two kinds are given for solutions of an elliptic equation in a given domain, estimates for arbitrary solutions on interior subdomains (interior estimates) and estimates on the whole domain for solutions of a Dirichlet problem (boundary estimates).

Let A be a subdomain of a fixed bounded domain of Euclidean n-space, $\acute{A}$ and $\bar{A}$ its boundary and closure respectively, D^ju any derivative of u of order j. If j is a non-negative integer, $0\le\alpha<1$, we designate by $C_{j+\alpha}{}^A$ the Banach space of functions on A with α-Hölder-continuous derivatives in $\bar{A}$ up to and including order j with

$$\|u\|_{j+\alpha}{}^A=\sum_{k\le j}\Big\{\sup_{x\in A}|D^ku(x)|\Big\}+\sup_{x,y\in A}\frac{|D^ju(x)-D^ju(y)|}{|x-y|^\alpha}.$$

Further, let $H_{j,p}{}^A$ be the completion of C_j^A with respect to the norm defined by $\|u\|_j{}^p=\sum_{k\leq j}\int_A |D^k u|^p dx$.

Let L be a linear elliptic differential operator on A. For L of the second order, the author discusses the estimates of Schauder for solutions of the equation $Lu=f$: $\|u\|_{2+\alpha}{}^{A_1}\leq c_{A_1}(\|f\|_\alpha+\|u\|_0)$ for any compact subdomain A_1 of A, where the constant c_{A_1} depends upon the diameter of A, the distance of A_1 from $\dot{A}$, the constant of ellipticity, and the α-Hölder norms of the coefficients of L. A corresponding boundary estimate, i.e., the corresponding estimate for the whole of A, is stated for the solution of the Dirichlet problem for null boundary data. The author discusses the application of such a-priori estimates to the proof of existence of solutions, both in the linear and non-linear cases. The latter is illustrated with the improvements that have been made in the above estimates for the special case of plane domains ($n=2$) by Morrey [Trans. Amer. Math. Soc. 43 (1938), 126–166], Nirenberg [Comm. Pure Appl. Math. 6 (1953), 103–156, 395; MR 16, 367] and Bers and Nirenberg [Convegno Internaz. sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, Edizioni Cremonese, Roma, 1955, pp. 111–140, 141–167; MR 17, 974].

Section 2 begins with a brief discussion of the L_2 existence theory for the Dirichlet problem for linear elliptic equations of higher order due to Višik, Gårding, and the reviewer. A somewhat generalized form is given to the definitions of elliptic and strongly elliptic system of differential operators. Boundary estimates for $H_{j,2}{}^A$ for solutions of the Dirichlet problem are described which were obtained independently by O. V. Guseva [Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 1069–1072; MR 17, 161], the author [Comm. Pure Appl. Math. 8 (1955), 649–675; MR 17, 742] and the reviewer [see the paper cited above]. Estimates in the interior of the Schauder type have been published for second-order systems by Morrey [Contributions to the theory of partial differential equations, Princeton, 1954, pp. 101–159; MR 16, 827] and for the general case by Douglis and Nirenberg [Comm. Pure Appl. Math. 8 (1955), 503–538; MR 17, 743]. The author states some results on interior estimates for the $(H_{j,p}{}^{A_1})$-norms using results of Calderón and Zygmund [Acta. Math. 88 (1952), 85–139; MR 14, 637] on singular integral operators in Euclidean spaces. These estimates have the significant feature that they use only the continuity of the highest-order coefficients of the operator L but no Hölder continuity. The main part of the paper concludes with the statement of some general results on the structure of solutions of general elliptic equations at isolated singularities with weak assumptions on the coefficients of the equation. The appendix contains the proof of an inequality, useful in deriving pointwise estimates from integral estimates on functions, which combines the well-known Sobolev and Morrey inequalities.

F. Browder (New Haven, Conn.).

Thyssen, M. Comportement asymptotique des fonctions et valeurs propres du problème de Dirichlet-Neumann pour $-\Delta+z$. Bull. Soc. Roy. Sci. Liège 25 (1956), 280–292.

The eigenvalues and eigenfunctions of the Laplace differential operator are considered on a smooth bounded and connected open set Ω of E^n with a null Dirichlet boundary condition on one portion of the boundary and a null Neumann condition on the remainder of the boundary. Let u_k be the kth eigenfunction of $-\Delta$ chosen from a complete orthonormal set in $L^2(\Omega)$ ordered with non-decreasing eigenvalues z_k. The author investigates the asymptotic behaviour as $z\to\infty$ of

$$\sum_{z_k\leq z} 1=N(z) \text{ and } \sum_{z_k\leq z} u_k(x)\overline{u_k(x_0)} \quad (x,\, x_0 \text{ in } \Omega).$$

The method employed is that of Carleman [Åttonde Skandinaviska Matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 34–44] which deduces the asymptotic behaviour by applying a Tauberian theorem to asymptotic estimates of the Green's function for $(-\Delta+z)$, z positive, with the mixed Dirichlet-Neumann boundary condition. General results have been obtained with this method by Minakshisundaram and Pleijel [Canad. J. Math. 1 (1949), 242–256; MR 11, 108], Gårding [Math. Scand. 1 (1953), 237–255; Kungl. Fysiogr. Sällsk. i Lund Forh. 24 (1954), no. 21; MR 16, 366; 17, 158], the reviewer [C. R. Acad. Sci. Paris 236 (1953), 2140–2142; MR 15, 320], and Ehrling [Math. Scand. 2 (1954), 267–285; MR 16, 706].

Using a Tauberian theorem of Hardy-Littlewood and some estimates of H. G. Garnir for the fundamental solution of $(-\Delta+z)$, the author proves along the lines of the first of the papers of Gårding mentioned above that

$$\lim_{z\to\infty} z^{-\frac{1}{2}n} \sum_{z_k\leq z} u_k(x)\overline{u_k(x_0)}=\delta_{x,x_0} \frac{1}{(4\pi)^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+1)},$$

where $\delta_{x,x_0}=0$ for $x\neq x_0$, $=1$ for $x=x_0$.

While the eigenfunction distribution was already obtained by Gårding, the asymptotic distribution for the eigenvalues,

$$z^{-\frac{1}{2}n}N(z) \underset{z\to+\infty}{\to} \frac{\text{meas }(\Omega)}{(4\pi)^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+1)},$$

was not. However, this part of the author's proof seems to be incomplete, the gap being embodied in the following statement on page 291 which is not justified by the previous arguments: "... en vertu du théorème de H. Lebesgue car

$$z^{-\frac{1}{2}n}\Big(\sum_{z_k\leq z} |u_k(x)|^2\Big)\leq \frac{1}{(4\pi)^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+1)} \in L_1(\Omega)."$$

What is necessary to complete the proof are estimates which imply that some iterate of the Green's function is a Hilbert-Schmidt kernel. (For a general class of elliptic boundary-value problems including the one considered here, this fact follows from results of the reviewer, Comm. Pure Appl. Math. 9 (1956), 351–361 [MR 19, 862].

F. Browder (New Haven, Conn.).

Migliau, Maria Giovanna. Sui problemi di Dirichlet e di Neumann per una semiellisse. Ricerche Mat. 6 (1957), 49–66.

Ghizzetti [Rend. Sem. Mat. Univ. Padova 20 (1951), 224–248; MR 12, 826] gave explicit solutions of the Dirichlet and Neumann problems for the ellipse in terms of Fourier series and the conformal transformation which carries the ellipse, slit between the foci, into an annulus. In deriving these solutions, he had to overcome the difficulty that in the transformed problem no boundary values are known on one of the two circular boundaries; this he did by noting that, on this boundary, the function must be even with respect to the angle variable θ, and the normal derivative must be odd, in order that, when the solution is transformed back, it should be harmonic across the slit. In the present paper, the same problems are explicitly solved for semi-ellipses, and by the same methods. The cases of the semi-ellipse bounded by major and minor axes are quite different; in the latter case, the difficulty mentioned above in connection with Ghizzetti's

problem must be overcome, while in the former the transformed problem is an ordinary boundary value problem for half an annulus. *J. W. Green.*

Laasonen, Pentti. On the behavior of the solution of the Dirichlet problem at analytic corners. Ann. Acad. Sci. Fenn. Ser. A. I. no. **241** (1957), 13 pp.

Let D be a plane domain whose boundary contains two closed analytic arcs C_1 and C_2 that meet at $z=0$ at an interior angle $\alpha\pi$ $(0<\alpha\leqq 2)$. Let f be the solution of a Dirichlet problem for Laplace's equation in D with boundary values on C_1 and on C_2 that are (except at $z=0$) two analytic functions of the arc length, regular on C_1 and C_2, respectively. Using a result of Lichtenstein [J. Reine Angew. Math. **140** (1911), 100–119] on conformal mapping and appraisals based on Cauchy's integral formula, the author proves that the nth partial derivatives of f near the corner $z=0$ are of order $O(|z|^{-n})$. An improvement of this result when f is continuous at $z=0$ is given. [These results are also immediate consequences of the asymptotic expansion for u derived by the reviewer in the Duke Math. J. **24** (1957), 47–56; MR **18**, 568.] *W. Wasow.*

Laasonen, Pentti. On the degree of convergence of discrete approximations for the solutions of the Dirichlet problem. Ann. Acad. Sci. Fenn. Ser. A. I. no. **246** (1957), 19 pp.

Let $f(x, y)$ be the solution of a Dirichlet problem for Laplace's equation in a bounded domain D with boundary B. Denote by $f_h(x, y)$ a solution of the simplest 5-point difference equation in a square net with mesh length h that approximates Laplace's equation formally, and assume that $f_h(x, y)$ approximates the boundary values of u to the order $O(h)$. The classical method for finding the order of magnitude of the truncation error $\delta_h=f_h-f$, as developed by Gerschgorin [Z. Angew. Math. Mech. **10** (1930), 373–382], shows that $\delta_h=O(h)$, provided f has bounded third derivatives in $D+B$. The author extends this method to problems whose boundary data are piecewise analytic. In these, the derivatives of f may be unbounded at the corners of D. The order of the nth derivatives of f at distance r from such a corner is $O(r^{-n})$, as $r\to 0$. This follows, for instance, from the results of the paper reviewed above. Gerschgorin's method was based on the construction of a suitable superharmonic comparison function which, by virtue of the maximum principle, yielded an upper bound for δ_h. The author determines a more complicated comparison function, unbounded at the corners, which enables him to show that Gerschgorin's result remains true in the interior of D, provided all corners of D are convex. However, if the largest interior angle at a corner of D is $\pi\alpha$, $\alpha\geqq 1$, then only the weaker result $\delta_h=O(h^{1/\alpha-\epsilon})$ $(\epsilon>0$, arbitrary) is obtained. Extensions to more refined difference approximations at the boundary and to boundary data that are piecewise in C^5 are included. *W. Wasow.*

Keller, J. B. On solutions of $\Delta u=f(u)$. Comm. Pure Appl. Math. **10** (1957), 503–510.

Let $h'(u)$ be a positive non-decreasing continuous function, defined for all real u, such that $\int_0^\infty [\int_0^x h'(z)dz]^{-\frac{1}{2}}dx<\infty$, and let $R(P)$ be the distance from a point P of the domain D of Euclidean n-space to the boundary S of D. The principal theorem states that there exists a decreasing function $g(R)$ such that if $u(P)$ is a solution of the equation $\Delta u=f(u)$ (where $f(u)$ is a real single-valued continuous function, defined for all real u, and $f(u)\geqq h'(u)$) then $u(P)\leqq g[R(P)]$; moreover, $g(R)\to\infty$ as $R\to 0$ and $g(R)\to-\infty$ as $R\to\infty$. It follows that u cannot be entire; the author notes that this fact is also given by R. Osserman [Appl. Math. Statist. Lab., Stanford Univ., 1956.] If also $f(u)$ is non-decreasing and D is bounded, then there exists a u which becomes infinite on S. Symmetric solutions of the differential equations are also considered for arbitrary real continuous $f(u)$ which are defined for all real u; a necessary and sufficient condition that such a solution be entire is given. This is related to a result announced by R. M. Redheffer [Bull. Amer. Math. Soc. **62** (1956), 408]. *F. W. Perkins* (Hanover, N.H.).

Miranker, W. L. The L^2-maximum principle for solutions of $\Delta u+k^2u=0$ in unbounded domains. Ann. of Math. (2) **67** (1958), 72–82.

Complex-valued solutions u of the differential equation in the title are studied in the exterior of an n-dimensional bounded domain, under the assumption that u satisfies the radiation condition:

$$\lim_{R\to\infty}\int_{\Sigma_R}|\partial u/\partial r-iku|R^{n-1}d\omega=0,$$

where Σ_R is the surface of some sphere of radius R with fixed center, and $d\omega$ is the surface element of the unit sphere. The constant k may be real or complex. Using methods similar to those derived by Vishik [Uspehi Mat. Nauk (N.S.) **6** (1951), no. 2(42), 165–166; MR **13**, 235] for harmonic functions, the author obtains bounds for the L^2-norm of the normal derivatives of u on the boundary, in terms of the boundary values of u and of its tangential derivatives. Since u can be represented as a sum of integrals involving u, its normal derivative, and a fundamental solution of the differential equation, there follows immediately a pointwise estimate of $|u(P)|$, which is the maximum principle referred to in the title. *W. Wasow.*

Lavruk, B. R. On regular solutions of boundary problems for elliptical systems of linear differential equations of the second order for a half-plane. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 107–111. (Ukrainian. Russian and English summaries)

Let $A(\partial/\partial x)$ denote the differential operator of elliptic type

$$A(\partial/\partial x)=A_{11}\partial^2/\partial x_1^2+2A_{12}\partial^2/\partial x_1\partial x_2+A_{22}\partial^2/\partial x_2^2,$$

whose coefficients are constant matrices of order p. Consider the boundary problem

$$A(\partial/\partial x)u(x)=0,\ x_2>0,$$
$$\lim_{x_2\to+0}(\partial^t/\partial x^t)B(\partial/\partial x)u(x)=0,$$

where $B(\partial/\partial x)=B_1(\partial/\partial x_1)+B_2(\partial/\partial x_2)$, with constant matrices B_t. The author proves, making certain assumptions, the uniqueness of the solution in the class of functions vanishing at infinity and also the continuous dependence of such a solution on the coefficients of the operators $A(\partial/\partial x)$ and $B(\partial/\partial x)$. Similar results are obtained for the boundary condition $\lim_{x_2\to+0}u(x)=f(x_1)$.

Ganin, M. P. The Dirichlet problem for the equation

$$\Delta U+\frac{4n(n+1)}{(1+x^2+y^2)^2}U=0.$$

Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5(77), 205–209. (Russian)

A method of obtaining a fairly explicit form for the solution of the problem of the title is described for the

case in which the region of solution is the unit circle with center at the origin. Starting from an integral representation, in terms of a Legendre polynomial and an arbitrary holomorphic function $\varphi(z)$, of an arbitrary regular solution of the equation for this region [given in I. N. Vekua, New methods for solving elliptic equations, OGIZ, Moscow-Leningrad, 1948; MR **11**, 598], the author uses classical methods to find $\varphi(z)$ by the imposition of the boundary condition. The different arbitrary terms in the solutions that are obtained in the two cases n odd, n even, are described and, in the course of the arguments, necessary conditions, in terms of the boundary function, that the problem have a solution are derived. *J. Cronin.*

Tersenov, S. A. An elliptical type of equation degenerating at the domain boundary. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 670–673. (Russian)

This is an extension of a paper by M. V. Keldyš [same Dokl. (N.S.) **77** (1951), 181–183; MR **13**, 41]. Let D be a region in the xy-plane bounded by a segment AB of the x-axis and a smooth arc Γ in the upper half-plane. The Dirichlet problem in D is studied, for (*) $yu_{yy}+u_{xx}+au_y+bu_x+cu=0$, where a, b, c are analytic functions of x and y, and $c\leqq 0$ in a subset of the upper half-plane which contains $\bar{D}$. {The J which appears in equation (1) of the paper is a misprint for y.} First it is shown that the function $\omega(x, y)=\int_y^1 \exp[\int_t^1 a(x, r)r^{-1}dr]dt+C_0$, where C_0 is a constant chosen so that $\omega>0$ on $\bar{D}$, has singularities at those points of AB where $a(x, 0)\geqq 1$. Theorem 1: Let $f(x, y)$ be a continuous function defined on $\Gamma+AB$. Then there exists a solution $u(x, y)$ of (*) in D, which is twice continuously differentiable and which satisfies the condition: $\lim_{(x,y)\to Q} u(x, y)/\omega(x, y)=f$ for $Q\in\Gamma+AB$.

Let $G=[(x, 0)\in AB|\lim_{y\to 0}\omega(x, y)=\infty]$, and let $G_0=\Gamma+AB-G$. Theorem 2: If $u(x, y)$ is a twice continuously differentiable solution of (*) in D and its boundary values are continuous on $\bar{G}_0$, and if $\lim_{y\to 0} u/\omega=0$ on G, then $u(x, y)$ is bounded on $\bar{D}$, and is uniquely determined by the given values on Γ and on those points of AB where $a(x, 0)<1$. *J. Cronin* (New York, N.Y.).

Levina, S. N. On the solution of the oscillation equation over the entire time axis. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 18–20. (Russian)

Given the equation (*) $u_{tt}-u_{xx}-u_{yy}-cu=0$ in the set $x\geqq 0$, $-\infty<y<\infty$, $-\infty<t<\infty$, the problem is to find a solution u of (*) such that $u|_{x=0}=f(t, y)$, a given function, and such that (condition B), given $\eta>0$, there is a $X(\eta)$ such that if $x>X(\eta)$ then $|u(x, y, t)|<\eta$. A formal solution is obtained by using a two-sided Laplace transformation (Mellin transformation), and a solution of (*) is obtained from this formal solution by imposing conditions on $f(t, y)$ sufficient to guarantee that the integrals in the formal solution converge, and that condition B is satisfied. {In the third inequality of condition A, $|f(t, y)|$ should be replaced by $|f_y'(t, y)|$.} Finally, a solution for the equation $u_{tt}-u_{xx}-u_{yy}-u_{zz}=0$ is briefly described. *J. Cronin.*

★**Weinstein, A. Elliptic and hyperbolic axially symmetric problems.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 264–269. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

This lecture presents a brief review of various problems concerning the partial differential equation $\partial^2u/\partial y^2+ky^{-1}\partial u/\partial y=L(u)$, where $u=u(x_1, \cdots, x_m, y)$; L is a linear differential operator in the variables $x_1, \cdots, x_m$ with constant or variable regular coefficients; and k is a real parameter, with $-\infty<k<+\infty$. The main concern is with the two cases in which L is either $-\Delta$ or $+\Delta$, where $\Delta u=\sum_{i=1}^m \partial^2u/\partial x_i^2$. In the first case the equation is elliptic, and one has "generalized axially symmetric potential theory" [Weinstein, Bull. Amer. Math. Soc. **59** (1953), 20–38; MR **14**, 749]. In the second case the equation is the hyperbolic "Euler-Poisson-Darboux" equation [Weinstein, C. R. Acad. Sci. Paris **234** (1952), 2584–2585; MR **14**, 176]. *J. B. Diaz.*

Diaz, J. B. On singular and regular Cauchy problems. Comm. Pure Appl. Math. **9** (1956), 383–390.

The author presents an excellent account of some of the work initiated by A. Weinstein and carried on by Weinstein, the author, and their colleagues on the singular equation $\Delta u=u_{tt}+ku_t/t$ and related equations. In particular, an example due to R. Davis of a solution of $u_{xx}=u_{tt}-2u_t/t$ with u given and $u_t=0$ on $t=t_0>0$ is given which does not converge to a solution as $t_0\to 0$.
H. F. Weinberger (College Park, Md.).

Vahaniya, N. N. A boundary problem for a hyperbolic system equivalent to the string vibration equation. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 906–909. (Russian)

In same Dokl. **109** (1956), 707–709 [MR **18**, 215] S. L. Sobolev solved the following problem for the square. The present author applies Sobolev's method to the rectangle with rational ratio for side-lengths and then develops a method of his own for irrational ratio. In the rectangle: $0\leqq x\leqq X$, $0\leqq t\leqq T$, with boundary Γ, find a solution of the system $\partial u_1/\partial x=\partial u_2/\partial t$, $\partial u_1/\partial t=\partial u_2/\partial x$ with boundary condition $au_1|_\Gamma+bu_2|_\Gamma=f$, where a, b and f are given on Γ.

John, Fritz. Non-admissible data for differential equations with constant coefficients. Comm. Pure Appl. Math. **10** (1957), 391–398.

Let P be a partial differential operator with constant coefficients and principal part P' and assume that P is weakly hyperbolic with respect to a real vector ξ, i.e. that P' is hyperbolic with respect to ξ. Although Cauchy's problem with data on the plane $\xi x=0$ may not be correctly set for P, it is true (Hörmander, unpublished) that there are infinitely differentiable solutions $\neq 0$ of $Pu=0$ whose Cauchy data have compact supports. It is shown that if P is any operator with $P'(\xi)\neq 0$ which has a solution with this property, then P has a weakly hyperbolic factor Q for which $Qu=0$. *L. Gårding* (Lund).

Pini, Bruno. Sul problema fondamentale di valori al contorno per una classe di equazioni paraboliche lineari. Ann. Mat. Pura Appl. (4) **43** (1957), 261–297.

This paper concerns the initial-boundary value problem for the parabolic equation

$$\mathscr{L}[u]=\partial u/\partial y+\sum_{i=1}^4 a_i(x, y)(\partial^i u/\partial x^i)=f(x, y)$$

for a plane domain $\mathscr{D}$ bounded by the lines $y=0$ and $y=1$ and the curves $x=X_i(y), i=1, 2, X_1(y)<X_2(y)$; $a_4(x, y)>0$ in $\bar{\mathscr{D}}$. The conditions to be satisfied are $u(X_i(y), y)=\varphi_i(y)$, $\partial u/\partial x(X_i(y), y)=\psi_i(y)$, $i=1, 2$, and $u(x, 0)=\omega(x)$.

The problem is first solved explicitly for the special parabolic operator $\mathscr{L}_0[u]=\partial u/\partial y+\partial^4u/\partial x^4$ by means of integral representations involving the fundamental solution of $\mathscr{L}_0$. Some interesting a-priori bounds are then established for solutions of $\mathscr{L}[u]=f$ under various differ-

entiability and Hölder-type assumptions on the boundary values of u, of f, and of the curves X_1 and X_2, and on the coefficients of $\mathscr{L}$. Use of the parametric method of E. E. Levi and of the explicit representation of the solution of the special problem finally leads to an integral equation for the solution of the desired problem.

As the author notes, the conditions imposed upon the data, coefficients, and boundary might be weakened; they can be. In particular, the conditions required for the author's uniqueness proof can be weakened considerably; e.g. with the use of inequalities analogous to the Gårding type for elliptic equations.
C. R. DePrima (Troy, N.Y.).

Zagorskiĭ, T. Ya. Quelques problèmes aux limites pour les systèmes des équations différentielles du type parabolique dans un demi-espace. Ukrain. Mat. Ž. **9** (1957), 252–270. (Russian. French summary)

Dans cet article on examine les systèmes des équations différentielles du type parabolique $\partial u/\partial t = A(\partial/\partial x)u$. On suppose que le système est aux coefficients constants et qu'il est homogène relativement à la dérivation d'après $x_1, x_2, \cdots, x_n$. L'auteur construit la résolution du système qui existe pour $x_n>0$, et satisfait aux conditions initiales $u|_{t=0}=0$ et aux conditions aux limites $\lim_{x_n\to 0} B(\partial/\partial x)u = f(x_1, \cdots, x_{n-1}, t)$. *Résumé de l'auteur.*

★ **Barenblatt, G. I. On self-similar solutions of the Cauchy problem for the nonlinear parabolic equation of the nonsteady filtration of a gas in a porous medium.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 6 pp.

Translation of Prikl. Mat. Meh. **20** (1956), 761–763 [MR **19**, 208].

Chen, Chin-i. A theorem on the uniqueness of the solution of a mixed problem for systems of linear partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 508–511. (Russian)

Etude d'un problème mixte pour le système

(1) $\quad \partial u/\partial t = P(t, \partial^n/\partial x^n)u,\ t\geqq 0,\ x\geqq 0,\ n$ entier $\geqq 2$,

$u=\{u_1, \cdots, u_N\}$, P=matrice différentielle.

On cherche u, solution de (1), avec $u(x,0)=u_0(x)$ donné et $\partial^{nl_j+m}u_j(0,t)/\partial x^{nl_j+m}=0$, $l_j=0, 1, \cdots, q_j-1$ (nq_j=ordre maximum de dérivation en x sur u_j), m entier fixé $\in \{0, 1, \cdots, n-1\}$. Si l'on impose à u une condition de croissance (exponentielle) à l'infini, uniformément en t, cet ordre étant lié aux propriétés de P, il y a au plus une solution. La démonstration utilise les espaces de Gelfand et Silov [Uspehi Mat. Nauk (N.S.) **8** (1953), no. 6(58), 3–54; Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 1065–1068; MR **15**, 867; **18**, 736; **17**, 267] et un résultat de Silov [ibid. **102** (1955), 893–895; MR **17**, 351].
J. L. Lions (Nancy).

Fridlender, V. R. On the problem of Cauchy-Kovalevski for certain partial differential equations. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 385–388. (Russian)

Extending some work of a previous paper [V. R. Fridlender and G. S. Salehov, Uspehi Mat. Nauk (N.S.) **7** (1952), no. 5(51), 169–192; MR **15**, 227], the author considers a more general equation

(*) $\quad \partial^p u/\partial t^p = A(t, x)\, \partial^q u/\partial x^q + f(t, x)$,

where $p>q$, in the region D ($|t|<R$, $x\in\mathfrak{M}$), where R is a given positive constant and $\mathfrak{M}$ is a given set; functions A, f are analytic in t and, as functions of x, belong to Gevrey's class $\delta=p/q$. Given the Cauchy data $\partial^k u/\partial t^k]_{t=0}=\varphi_k(x)$ ($x\in\mathfrak{M}$; $k=0, 1, \cdots, p-1$), then, in order that (*) have a solution satisfying this initial data and analytic in t for t close to $t=0$, it is sufficient that the given functions $\varphi_k(x)$ ($k=0, 1, \cdots, p-1$) be of class δ in $\mathfrak{M}$. The solution is unique, and if A and f are entire functions of t, and A, f, φ_k ($k=0, 1, \cdots, p-1$) belong to class $\alpha<\delta$ in $\mathfrak{M}$ for arbitrary t, then the solution is an entire function of t for each $x\in\mathfrak{M}$. Generalizations to more complicated equations and systems of equations are indicated. *J. Cronin* (New York, N.Y.).

See also: Ordinary Differential Equations: Delsarte and Lions; Chan. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Fichera; Nelson. **Manifolds, Connections:** Narasimhan. **Elasticity, Plasticity:** Singh. **Fluid Mechanics, Acoustics:** Coburn.

Difference Equations, Functional Equations

Bajraktarević, M. Sur une solution monotone d'une équation fonctionnelle. Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 43–52.

The paper deals with the functional equation (*) $F(z)=\varepsilon_0 f\{z, F[g(d_0, z)]\}$ for $z\in J=[a_1, a_2]$ and $t_0^{(1)}<F(z)<t_0^{(2)}$. Here ε_0 is a constant, and $f(z,t)$, $g(d_0, z)$ are given functions. $f(z,t)$ is assumed to be continuous and strictly increasing in z and t for $z\in J$, $-\infty<t<\infty$, and in addition f and g are subject to further conditions too lengthy to state here. It is shown that equation (*) has one and only one strictly monotone solution $F(z)$, defined almost everywhere on J and with values on the interval $(t_0^{(1)}, t_0^{(2)})$. Equation (*) is related to a functional equation considered (by an entirely different method) by A. H. Read [Proc. Roy. Soc. Edinburgh. Sect. A. **63** (1952), 336–345; MR **14**, 286]. *I. M. Sheffer.*

See also: General Theory of Numbers: Johnson. **Analytic Theory of Numbers:** Bochner.

Integral and Integrodifferential Equations

See: Banach Spaces, Banach Algebras, Hilbert Spaces: Granas; Kalisch; Sahnovič.

Calculus of Variations

Beckert, Herbert. Existenzbeweise mehrdimensionaler regulärer Variationsprobleme. Math. Ann. **133** (1957), 191–218.

The present paper is concerned with existence theorems for multiple integral problems in the calculus of variations. The particular integral to be minimized is of the form

$$I=\int_B f(x, u, p)dx,$$

where $x=(x_1, \cdots, x_m)$, $u=(u_1, \cdots, u_n)$ and p is the matrix whose elements are the partial derivatives $p_{i\beta}=\partial u_i/\partial x_\beta$. For convenience, the author restricts himself mainly to the case $m=2$ and n arbitrary and to the case of fixed boundaries. Under the assumption that the Legendre condition holds in its strongest form and that

a certain strengthened Jacobi condition holds, it is shown that there are absolutely continuous functions u_i in the sense of Tonelli taking on prescribed values on the boundary of E which minimize I. For the case when f is quadratic in u_i and $p_{i\beta}$ this result is valid under the assumption that the coefficients are bounded and measurable in x on E. The paper represents a notable contribution to existence theorems and differentiability theorems for multiple integral problems and relates the results obtained with those previously obtained by the author and by others. *M. R. Hestenes* (Los Angeles, Calif.).

Siegel, Carl Ludwig. Integralfreie Variationsrechnung. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. **IIa.** 1957, 81–86.

The paper is concerned with proving the existence of a solution of the calculus of variations problem associated with a function $F(x^1, \cdots, x^n; \dot{x}^1, \cdots, \dot{x}^n)$, with very general conditions upon the function F. The chief feature is the way in which the usual integration by parts used in obtaining the Euler equations is avoided, and in fact no use is made of the integral idea.

The existence of rectifiable minimizing arcs had been proved under exceedingly general conditions by Hilbert. To establish differentiability properties of the solutions, one uses the Euler equations, assuming therefore the existence of second partial derivatives of $F(x, \dot{x})$.

The question of what differentiability properties of the solutions are due to which properties of $F(x, \dot{x})$ was examined by Busemann and Mayer [Trans. Amer. Math. Soc. **49** (1941), 173–198; MR **2**, 225]. They proved that the minimizing arcs are continuously differentiable, if one has continuity, strict convexity and a Lipschitz condition, without assuming even the existence of first partial derivatives of $F(x, \dot{x})$. But their method only applies to two-dimensional spaces. The author, with a different set of conditions of a very general nature, establishes the corresponding result for dimension $n>2$.

E. T. Davies (Southampton).

TOPOLOGICAL ALGEBRAIC STRUCTURES

See: Fields, Rings: Jans.

Topological Groups

Wright, Fred B. Hölder groups. Duke Math. J. **24** (1957), 567–571.

According to a long established theorem of Hölder, any Archimedean linearly ordered group is isomorphic to a subgroup of the additive group of real numbers. In particular, such a group is an Abelian group.

There are several ways of generalizing this theorem. The usual procedure is to assume a partial order rather than a linear order, and to require of the group a certain behavior relative to the partial order. For example, the existence of order units is frequently assumed. The additive group of real numbers is replaced by a suitable generalization, say by a function space or by a partially ordered vector space of a more general nature.

The purpose of this note is to offer another sort of generalization. In this approach, all a priori reference to ordering is suppressed. This casts the entire burden of the argument on the group structure. The assumption of an ordering in the group is replaced by a maximality condition, and the more restrictive hypothesis of Archimedean type is replaced by a correspondingly stronger maximality condition. The ordering also disappears from the groups used in the representation theory. In their place appear locally convex linear spaces. (From the introduction.) *D. Montgomery* (Princeton, N.J.).

Numakura, Katsumi. Prime ideals and idempotents in compact semigroups. Duke Math. J. **24** (1957), 671–680.

Let S denote a Hausdorff topological semigroup, and let M be a (two-sided) ideal of S. A subset A of S is said to be nilpotent with respect to M if for any open U about M there is a positive integer n_0 such that $n \geqq n_0$ implies $A^n \subset U$. A subset B of S is called a nil subset with respect to M if B consists of nilpotent elements with respect to M. The radical of M in S is defined to be the largest nil ideal of S with respect to M. If S has a minimal ideal K, then the radical of S is defined to be the radical of K in S, and is denoted by N. Nilpotence with respect to K is termed briefly nilpotence. A left (right or 2-sided) ideal P of S is said to be prime if for any two-sided ideals A and B of S with $AB \subset P$ one has $A \subset P$ or $B \subset P$.

It is shown that if S is compact and $S \neq K$, then N coincides with the intersection of open prime ideals of S and also with the union of all nilpotent ideals (whether left, right, or 2-sided).

For $a \in S$, let $J_0(S \setminus a)$ denote the union of all two sided ideals not containing a. If S is compact, it is shown that the family of all open prime ideals $\neq S$ coincides with the family of ideals $\{J_0(S \setminus e)\}$ where e runs over the idempotents in $S \setminus K$.

Finally it is shown that if S is compact and if $\{x \mid SxS = SeS\}$ is a group for each idempotent e of S, (this is equivalent to the author's conditions (R) and (L)) then the set of open prime ideals is a lattice isomorphic to E (the set of idempotents) with a maximal element adjoined. Under these conditions (which, the author does not remark, are realized in the abelian case), if $S \neq K$, then the set of all nilpotent elements of S coincides with the radical of S; that is, if a is nilpotent and $x \in S$, then ax and xa are nilpotent. *R. J. Koch* (Baton Rouge, La.).

See also: Approximations, Orthogonal Functions: Morgenthaler; Fine.

Topological Vector Spaces

Raĭkov, D. A. On a property of nuclear spaces. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 5(77), 231–236. (Russian)

A topological linear space E is countably normed if its topology is given by a non-decreasing sequence of norms $\|\cdot\|_1 \leqq \|\cdot\|_2 \leqq \cdots$ having the property that $\|x_k\|_{n+1} \to 0$ whenever $\{x_k\}$ is fundamental for $\|\cdot\|_{n+1}$ and $\|x_k\|_n \to 0$. A linear transformation T from a Banach space X into a Banach space Y is nuclear if there exist sequences $\{f_i\}$ in X^* and $\{y_i\}$ in Y such that $\sum_{i=1}^{\infty} \|f_i\| \, \|y_i\| < \infty$, and $Tx = \sum_{i=1}^{\infty} f_i(x) y_i$, for each x in X. A countably normed space E is nuclear in the sense of Grothendieck if for each $n>0$ there exists $m>n$ such that the inclusion $E_m \to E_n$ is

nuclear, where E_k is the completion of E in the norm $\|\cdot\|_k$. A countably normed space E is nuclear in the sense of Gelfand if, whenever a sequence $\{f_i\}$ in E^* is such that $\sum_{i=1}^{\infty} |f_i(x)|<\infty$ for each x in E, then there exists $n>0$ such that f_i is in $E_n{}^*$ for each i and $\sum_{i=1}^{\infty} \|f_i\|_n<\infty$. The author proves that if a complete, countably normed space is nuclear in the sense of Grothendieck, then it is nuclear in the sense of Gelfand. *R. R. Phelps.*

Koshi, Shôzô. Modulars on semi-ordered linear spaces. II. Approximately additive modulars. J. Fac. Sci. Hokkaido Univ. Ser. I. 13 (1957), 166–200.

The author continues the study of non-additive modulars begun in Part I [M. Miyakawa and H. Nakano, same J. 13 (1956), 41–53; MR **19**, 664]. He discusses modulars with properties which are close to additivity, and also discusses uniform properties of general non-additive modulars. *I. G. Amemiya* (Kingston, Ont.).

Shimogaki, Tetsuya. On certain property of the norms by modulars. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1957), 201–213.

Let R be a modulared semi-ordered linear space with a modular m [see H. Nakano, Modulared semi-ordered linear spaces, Maruzen, Tokyo, 1950; MR **12**, 420]. It was known that for the two norms defined in R as $\|x\|=\inf\{1+m(\xi x)/\xi|\ \xi>0\}$ and $|||x|||=\inf\{1/|\xi|\ |m(\xi x)\leq 1\}$, the ratio $\alpha(x)=\|x\|/|||x|||$ for $x\neq 0$ varies from 1 to 2, and if $\alpha(x)$ is a constant, then R is essentially of L_p-type [S. Yamamuro, Proc. Japan Acad. **27** (1951), 623–624; MR **14**, 482; I. Amemiya, J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1954), 22–33; MR **17**, 387]. Now the author investigates the condition (*) that the infimum of $\alpha(x)$ is strictly more than 1, and proves that the condition: $f(\xi)<+\infty$ for all ξ, and $g(\xi)/\xi\to+\infty$ as $\xi\to+\infty$, where $f(\xi)=\sup\{m(x)|\ |||x|||=\xi\}$, and $g(\xi)=\inf\{m(x)|\ |||x|||=\xi\}$, is a sufficient condition for (*), and it is also necessary in case R is non-atomic. He also discusses the behaviours of the functions $f(\xi)/\xi^p$ and $g(\xi)/\xi^p$ in the neighbourhood of $\xi=0$ and $\xi=+\infty$, and their relations to the condition (*). *I. G. Amemiya.*

Chong, F. Schwartz's theory of distributions. Austral. J. Sci. **20** (1957), 1–4.

An expository paper intended for readers whose interest is primarily in applications.

Asplund, Edgar. A non-closed relative spectrum. Ark. Mat. 3 (1958), 425–427.

Let T be a continuous linear transformation of a topological vector space E into itself. A relative inverse of T is a continuous linear transformation R of E into itself such that $TRT=T$. The set of complex numbers λ such that $T-\lambda I$ does not have a relative inverse is called the relative spectrum of T. The author gives an example of a bounded linear transformation, defined on a Hilbert space, whose relative spectrum is not closed.
P. Saworotnow (Washington, D.C.).

See also: Ordinary Differential Equations: Delsarte and Lions. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Turumaru; Fichera; Kalisch. **Numerical Methods:** Altman.

Banach Spaces, Banach Algebras, Hilbert Spaces

Granas, A. Über eine Klasse nichtlinearer Abbildungen in Banachschen Raumen. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 867–871. (Russian. German summary)

The quasinorm of an (in general non-linear) operator on a Banach space is defined as $|H|=\inf \sup (|Hx|/|x|)$ where the inf is taken over all positive numbers r and the sup is taken over all x in the Banach space such that $|x|\geqq r$. An operator is quasibounded if its quasinorm is less than infinity. It is shown that if H (on a real Banach space to itself) is completely continuous and quasibounded and $|k||H|<1$ then the equation $y=x+kHx$ has at least one solution x for every y in the Banach space. This result is then applied to a class of nonlinear integral equations of the Hammerstein type. *D. C. Kleinecke.*

Granas, A. Über einen geometrischen Satz in Banachschen Räumen. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 873–877. (Russian. German summary)

The real Banach space X is the direct sum of two subspaces A and B, and P and Q are the projections of X onto A and B respectively. If F and G are completely continuous quasibounded [see preceding review] mappings from A and B into X, and $|F|\ |P|+|G|\ |Q|<1$, then the image of A under the mapping $I+F$ intersects the image of B under $I+G$. A condition is given for this intersection to consist of a single point. *D. C. Kleinecke.*

Altman, M. An intersection theorem in Hilbert space. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 963–966, LXXXI. (Russian summary)

The result of the preceding review is formulated and proved for weakly continuous operators (carrying weakly convergent sequences into weakly convergent sequences) on a separable real Hilbert Space. *D. C. Kleinecke.*

Turumaru, Takasi. On the direct product of operator algebras. IV. Tôhoku Math. J. (2) **8** (1956), 281–285.

M. Nakamura [same J. **6** (1954), 205–207; MR **16**, 1126] has discussed the relation between the direct product of and the ring generated by two sub-factors of a finite W^*-factor, and obtained a satisfactory analogue to the result in the classical theory of hypercomplex numbers: that is, in a finite W^*-factor, the direct product (in the W^*-sense) of two elementwise commutative sub-factors is the ring generated by them in the weak operator topology, and vice versa. In the present paper, we consider the same problem for subalgebras in an arbitrary C^*-algebra. {The earlier parts of this article were reviewed in MR **14**, 991; **15**, 237; and **16**, 1126.} *From the author's summary.*

Korenblyum, B. I. On a normed ring of functions with convolution. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 226–229. (Russian)

The author announces without proof further results on the algebra $L(\alpha)$ studied in an earlier communication [same Dokl. (N.S.) **111** (1956), 280–282; MR **19**, 46]. Notation and terminology are as in the review cited. For a non-positive real number γ, let $I_\gamma{}^+$ be the set of all $f\in L(\alpha)$ such that $\gamma^+(f)\leqq\gamma$; $I_\gamma{}^-$ is defined analogously. Theorem 1: $I_\gamma{}^+$ and $I_\gamma{}^-$ are closed primary ideals in $L(\alpha)$. $I_\gamma{}^+$ consists of all functions $f\in L(\alpha)$ such that

$$\int_{-\infty}^{\infty} f(x-y)\,\frac{\exp(-i\mu_1 x)dx}{\Gamma(\frac{1}{2}+2\alpha x i\pi^{-1})}=0 \quad (-\infty<y<\infty),$$

where $\mu_1=2\alpha\pi^{-1}\log|\gamma|$. A similar description is given for $I_\gamma{}^-$. Every closed primary ideal of $L(\alpha)$ has the form $I_{\gamma_1}{}^+\cap I_{\gamma_2}{}^-$.

For $\mathfrak{M}\subset L(\alpha)$, let the skeleton $\sigma(\mathfrak{M})$ of $\mathfrak{M}$ be the set of all complex numbers z, $|\mathrm{Im}\, z|\leqq\alpha$, such that the Fourier transforms of all $f\in\mathfrak{M}$ vanish at z. Theorem 2: Let $I\subset L(\alpha)$ be a closed ideal whose skeleton is a finite set $\{z_1, \cdots, z_k\}$

of interior points of the strip $|\mathrm{Im}\, z| \leq \alpha$. Let n_λ ($\lambda=1, \cdots, k$) be the largest number n such that all Fourier transforms of functions in I vanish to order n at z_λ. Let $\gamma_1 = \sup \gamma^+(f)$, $\gamma_2 = \sup \gamma^-(f)$, both suprema being taken over all $f \in I$. Then I consists of all functions belonging to $I_{\gamma_1}{}^+ \cap I_{\gamma_2}{}^-$ whose Fourier transforms have zeros of order at least n_λ at z_λ ($\lambda=1, \cdots, k$). This answers in the negative a question asked by Mackey [Bull. Amer. Math. Soc. **56** (1950), 385–412; MR **12**, 588]. A number of related results are also announced. *E. Hewitt.*

Šilov, G. E. Letter to the editor. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5(77), 270. (Russian)

The author withdraws his previous criticism [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 246–249; MR **18**, 912] of a proof given by Arens and Calderón.
E. Hewitt (Seattle, Wash.).

Vidav, Ivan. Über die Darstellung der positiven Funktionale. Math. Z. **68** (1958), 362–366.

Let $\mathfrak{B}$ be a linear complex algebra of elements $x, y, \cdots$, with unit element 1, on which an involution $x \to x^*$ of $\mathfrak{B}$ into itself is defined, satisfying $x^{**}=1$, $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$, and $(xy)^* = y^* x^*$. An element x is called symmetric if $x^* = x$. A linear functional $f(x)$ on $\mathfrak{B}$ is said to be positive if $f(x^*x) \geq 0$ for all x in $\mathfrak{B}$. Henceforth, $\mathfrak{B}$ will denote that particular algebra generated by two symmetric elements p and q satisfying the commutation relation $pq - qp = -i$ ($i^2 = -1$). If $p + iq = 2^{\frac{1}{2}} a$, $p - iq = 2^{\frac{1}{2}} a^*$, then $aa^* - a^*a = 1$. If $f(x)$ is a positive functional on this algebra satisfying $f(1)=1$, then the elements x of $\mathfrak{B}$ for which $f(x^*x)=0$ constitute a left ideal $\mathfrak{N}$. The Hilbert space $\mathfrak{H}$ is defined to be the completion of the Hilbert space, $\mathfrak{B}/\mathfrak{N}$, of residue classes mod $\mathfrak{N}$. Every element x in $\mathfrak{B}$ determines a linear operator A_x in $\mathfrak{H}$ by: $A_x\{x_0 + \mathfrak{N}\} = \{xx_0 + \mathfrak{N}\}$, where $\{x + \mathfrak{N}\}$ denotes the residue class containing x. The principal result can then be stated as follows. Let $f(x)$ be a positive functional on $\mathfrak{B}$ satisfying $f(1)=1$, and suppose that the closure of $A_{aa^*} = A_a A_{a^*}$ is self-adjoint. Then there exists a sequence of functions $\psi_k(q)$, $k=1, 2, \cdots, n \leq \infty$, each of class C^∞ on $-\infty < q < \infty$, such that, for all x in $\mathfrak{B}$, $f(x) = \sum_{k=1}^n \int_{-\infty}^{\infty} X\psi_k(q)\bar{\psi}_k(q)\,dq$. Here X is the differential operator $X = \sum \alpha_{km}(-i\,d/dq)^k q^m$ corresponding to the element $x = \sum \alpha_{km} p^k q^m$ of $\mathfrak{B}$ (α_{km} being complex numbers). *C. R. Putnam.*

Bădescu, Radu. Sur quelques équations fonctionnelles dans l'espace de Hilbert complet L^2. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 273–290. (Romanian. Russian and French summaries)

The problem of the effective determination of the eigenvalues of a linear transformation Θ on L_2 to L_2 is solved by means of the analytic continuation of the solutions of the functional equation $\Phi - \mu\Theta(\Phi) = \psi$, where Φ is an unknown element of L_2, the parameter μ is a complex number, the transformation Θ leaves invariant a certain subspace E of L_2, and ψ is a given element of E. Many applications are given, in particular to Schröder's functional equation $\Phi(z) = \mu\Phi(\alpha z)$. *J. B. Diaz.*

Glazman, I. M. An analogue of the extension theory of Hermitian operators and a non-symmetric onedimensional boundary problem on a half-axis. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 214–216. (Russian)

Let H be a Hilbert space, and J a (nonlinear) involution on H such that $(Jf, Jg) = (g, f)$ identically. A linear operator A with domain D_A is J-symmetric if $(Af, Jg) = (f, JAg)$, and is J-selfadjoint if D_A is dense and $JAJ = A^*$. In case $\mathrm{Im}(Af, f) \geq 0$, $f \in D_A$, A is called dissipative. [Cf. I. M. Glazman, Uspehi Mat. Nauk (N.S.) **5** (1950), no. 6(40), 102–135; MR **13**, 254.] Theorem 1: If A is closed, dissipative and J-symmetric then A is J-selfadjoint if and only if its deficiency index $m = \dim\{H \ominus (A - \lambda I)D_A\}$ vanishes. (m is constant for $\mathrm{Im}\,\lambda < 0$.) Theorem 2: If A with dense domain is J-symmetric and dissipative then A admits a J-selfadjoint extension.

In the application, $H = L_2(0, \infty)$ and A is defined through a differential operator

$$l[y] = \sum_{k=1}^{n} (-1)^k D^k [p_{n-k} D^k](y), \quad D = d/dx,$$

acting on functions y which have absolutely continuous derivatives to order $2n-1$, all of which vanish at zero; the p's satisfy "customary smoothness conditions", and if their imaginary parts are nonnegative A is dissipative. Three theorems generalizing Glazman's results are stated; when $n=1$ they also contain Weyl's classical results [Math. Ann. **68** (1910), 220–269] for the differential equation $-y'' + qy - \lambda y = g$, in which $\mathrm{Im}\, q$ is semibounded.
J. G. Wendel (Ann Arbor, Mich.).

★ **Fichera, Gaetano. Methods of functional linear analysis in mathematical physics: "a priori" estimates for the solutions of boundary value problems.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 216–228. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. \$7.00.

Let D be an open set of Euclidean space R_n, and let B_1 be a subset of the boundary B of D. Let S denote the space of all admissible functions u, and $L(u)$ be a linear operator which transforms u into a function defined on B_1. The following boundary value problem is considered: $E(u) = f$, in D; $L(u) = g$, on B_1; where $E(u)$ is a linear differential operator whose coefficients are defined in D. The author is concerned with the problem of giving upper and lower bounds for the unknown solution and its derivatives, in the following precise formulation. Suppose that, under a suitable definition of a norm, the space S of all admissible functions is a Banach space and that (f, g) is an element of a Banach space S'. Consider the linear transformation $T(u) = (E(u), L(u))$, which maps S on a linear subset of S'. The author's problem consists first, of showing that, under proper circumstances, there exists an optimum constant K such that $\|u\| \leq K\|T(u)\|$ for all u in S; and secondly, of determining an upper bound for such a constant K. In this paper a general procedure for doing this is outlined in connection with certain problems arising in applications; in particular, in second order parabolic and elliptic partial differential equations, and certain systems of linear partial differential equations. The underlying idea consists of associating with the transformation T and with the Banach space S a functional $F(w)$ and a class Γ of vectors w, satisfying the following conditions: (1) If K exists, then Γ is not empty; (2) for every u in S and every w in Γ, one has the inequality $\|u\| \leq F(w)\|T(u)\|$. For details of the proofs, reference is made to Fichera, Ann. Mat. Pura Appl. (4) **36** (1954), 273–296 [MR **16**, 374].
J. B. Diaz (College Park, Md.).

Nelson, Edward. Kernel functions and eigenfunction expansions. Duke Math. J. **25** (1957), 15–27.

The author establishes a general form of Mautner's eigenfunction expansion for self-adjoint operators the

projections of whose spectral resolutions are given by integral operators. The analysis proceeds through establishing a Radon-Nikodym theorem for positive-definite functions, and in terms of Aronszajn's notion of reproducing kernel. Necessary and sufficient conditions that a function of an operator of the type considered be an integral operator, and a generalization of Titchmarsh' formula for the projection operators in terms of integral of the Green's kernels, are given. *J. Schwartz.*

Kalisch, G. K. On similarity, reducing manifolds, and unitary equivalence of certain Volterra operators. Ann. of Math. (2) **66** (1957), 481–494.

The Volterra operator considered is of the form $T_F f=\int_x^1 F(x,y)f(y)dy$, mapping $L_p[0,1]$ with $1<p<\infty$ into itself. This is the continuous analog of n by n triangular matrices $M=m_{ij}$ with $m_{ij}=0$ for $i\geqq j$. The nilpotency of such matrices is paralleled in the continuous case by functions $F(x,y)=(x-y)^{m-1}G(x,y)$, with $G(x,x)\neq 0$. Only complex-valued functions of this type, m an integer (the order of F), $G(x,x)$ real and nonvanishing, are considered. Earlier treatment of such transformations is due to J. Pérès [J. Math. Pures Appl. (7) **1** (1915), 1–97] and V. Volterra and J. Pérès [Leçons sur la composition et les fonctions permutables, Gauthier-Villars, Paris, 1924]. If F is of order m, then T_F is shown to be similar to a unique operator $cT_E{}^m$ ($T_E{}^m$ the mth iterate of the operator T_E), where $cT_E{}^m=PT_FP^{-1}$. Here P is a product of linear transformations having reciprocals of one of the following three types: (1) multiplication by a measurable function; (2) substitution (change of measure on [0, 1]) via a monotone function $r(t)$ on [0, 1] onto [0, 1]; and (3) a linear transformation of the form $I+T_M$, where I is the identity, and $T_Mf=\int_x^1 M(x,y)f(y)dy$. The only reducing manifolds of T_F are the subspaces $L_p[0,c]$ for all c, $0\leqq c\leqq 1$, if (a) $F(x,y)\in C^2$ and of order 1; or (b) F is analytic in (x,y) and of order m, m a positive integer; or (c) $F(x,y)=F(y-x)=(y-x)^{m-1}k(y-x)+n(y-x)$, where F is of order m, $k\in C^2$ and $n\in L_1[0,1]$, vanishing in a vicinity of $y=x$. For the case where the basic class is $L_2[0,1]$, it is possible to obtain analogous results of equivalence under unitary linear transformations U: $T_1=UT_2U^*=UT_2U^{-1}$, where U is of the type (1) or (2) above. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Sahnovič, L. A. On reduction of Volterra operators to the simplest form and on inverse problems. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 235–262. (Russian)

M. S. Livšic has shown [Mat. Sb. N.S. **34(76)** (1954), 145–199; MR **16**, 48] that a linear operator of class $(i\Omega)$ in Hilbert space, with a spectrum consisting of 0 only, is unitarily equivalent to an operator of the form (1) $Af=i\int_0^x f(t)\beta(t)Jdt\cdot\beta(x)$, in the Hilbert space $L_r{}^2[0,l]$ which is a direct product of r copies of $L^2[0,l]$, r being finite or infinite; the matrix $\beta(x)$ is non-negative definite, and $\mathrm{tr}[\beta^2(x)]=1$ for all x; J is a diagonal matrix whose diagonal elements are $+1$ or -1.

In the present paper, the author considers the further simplification of (1) by similarity transformations. Suppose that one can write $V(x)\beta^2(x)JV^{-1}(x)=D(x)$ $(0\leqq x\leqq l)$, where $D(x)$ is a real diagonal matrix, all of whose non-zero elements have the same value, $d(x)$; where $V(x)$, $V'(x)$, $V''(x)$, $V^{-1}(x)$, $D(x)$, $D'(x)$ are bounded functions of x; and where $D(x)$ has non-zero rank for all x; then the simple part (in Livšic's sense) of A is similar to an operator $I^{(m)}$, where $I^{(m)}\varphi=[i\int_0^y\varphi_1(t)dt,\cdots,i\int_0^y\varphi_m(t)dt,0,0,\cdots]$, $0\leqq y\leqq\omega(l)=\int_0^l d(t)dt$, and the supplementary component of A is $\{f:f\in L_r{}^2,\ f(x)\beta(x)=0\ (0\leqq x\leqq l)\}$. Some modifications and generalizations of these results are given later in the paper.

In particular, let $Kf=i\int_0^x K(x,y)f(y)dy$ $(0\leqq x\leqq l)$ be an ordinary Volterra integral operator, where $K(x,y)=\sum_{j=1}^r \varepsilon_j\varphi_j(x)\overline{\varphi_j(y)}$, $\varepsilon_j=\pm1$; the φ_j are linearly independent; $\sum|\varphi_j|^2$, $\sum|\varphi_j'|^2$ and $\sum|\varphi_j''|^2$ are uniformly convergent; $\varphi_1(x)\neq 0$; and $K(x,x)\neq 0$. Then K is similar to an operator $I^{(1)}$. The author deduces that the only invariant subspaces of K are those of the form $H_\alpha=\{f:f(x)=0\ (0\leqq x\leqq\alpha)\}$.

As an application, the author shows that if $W_1(x,\lambda)$ and $W_2(x,\lambda)$ satisfy differential equations

$$\frac{dW_j(x,\lambda)}{dx}=-\frac{i}{\lambda}W_j(x,\lambda)\beta_j{}^2(x)J\quad(0\leqq x\leqq l;\ j=1,2),$$

where $W_j(0,\lambda)=I$, and the expressions $\beta_j{}^2(x)J$ satisfy the conditions described above, and if $W_1(l,\lambda)=W_2(l,\lambda)=W(\lambda)$, then there is a constant unitary matrix V such that $V\beta_1(t)V^{-1}=\beta_2(t)$, $VW(\lambda)V^{-1}=W(\lambda)$. Under more special conditions, it can be shown that $\beta_1{}^2(x)J=\beta_2{}^2(x)J$; from this, conditions are deduced under which a system of ordinary linear differential equations is completely defined by its Wronskian. *F. Smithies.*

See also: Measure, Integration: Gagliardo. **Sequences, Series, Summability:** Parameswaran; Favard. **Integral Transforms:** Fox. **Ordinary Differential Equations:** Lidskiĭ. **Topological Vector Spaces:** Asplund. **Numerical Methods:** Altman.

TOPOLOGY

General Topology

Kuratowski, K. Sur quelques invariants topologiques dans l'espace euclidien. J. Math. Pures Appl. (9) **36** (1957), 191–200.

In this paper, the author gives an elementary proof of the theorem that if A is a closed subset of the n-sphere, S^n, then any subset B of S^n which is homeomorphic to A has the property that S^n-A and S^n-B have the same number of components. This theorem, a classical one in topology, is an immediate corollary of the Alexander duality theorem, but in the present paper the author does not use the methods of algebraic topology. In addition the author proves several analogous theorems by elementary methods. These give some conditions on a map from A to B which guarantee the same theorem without demanding the map be a homeomorphism. *J. C. Moore.*

Rudin, Mary Ellen. A property of indecomposable connected sets. Proc. Amer. Math. Soc. **8** (1957), 1152–1157.

A connected set I is indecomposable if it is not the union of two connected sets, neither of which is dense in I. Theorem: Suppose I is an indecomposable connected subset of the plane and p is a limit point of I. Then $I\cup\{p\}$ is also an indecomposable connected set. The proof depends upon a complicated construction which makes

essential use of the properties of the plane. The author remarks that the above theorem is not true in 3-space.
R. Ellis (Philadelphia, Pa.).

Kubenskiĭ, A. A. Functionally-closed spaces. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 748–750. (Russian)

A discussion of Q-spaces (called functionally-closed spaces by the author). Everything stated in this note is found in the reviewer's original paper [Trans. Amer. Math. Soc. **64** (1948), 45–99; MR **10**, 126] or in Shirota's paper on the subject [Osaka Math. J. 4 (1952), 23–40; MR **14**, 395].
E. Hewitt (Seattle, Wash.).

Parovičenko, I. I. Certain special classes of topological spaces and δs-operations. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 866–868. (Russian)

The first part of the paper concerns spaces in which any intersection of less than $\aleph_\alpha$ open sets is open; these are called T^α-spaces, or T_i^α spaces if they are also T_i. A typical result is that a T_3^α-space in which every open covering has a subcover of power $\aleph_\alpha$ is a T_4^α-space. [For $\alpha=1$, this is in Gillman and Henriksen, Trans. Amer. Math. Soc. **77** (1954), 340–362, p. 350; MR **16**, 156.] The second part of the paper concerns anti-Hausdorff spaces (spaces without disjoint non-empty open sets). The results are too technical to be described here.
J. Isbell.

Parovičenko, I. I. On topological spaces whose weight exceeds their power. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 1074–1076. (Russian)

The weight of a space is the least cardinal number of a basis of open sets. The author points out that for each infinite cardinal m, the number of non-homeomorphic T_3-spaces (in fact, door spaces) of power m is greater than the number of T-spaces which have both power and weight $\leqq m$. He then constructs a number of examples and fills in some gaps in a proof of Yu. Smirnov [Izv. Akad. Nauk SSSR. Ser. Mat. **14** (1950), 155–178; MR **11**, 675].
J. Isbell (Seattle, Wash.).

Šersnëv, M. Characterization of the dimension of metric spaces by continuous mappings into Euclidean spaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5(77), 245–247. (Russian)

It is shown that Katětov's results on non-separable metric spaces [Czechoslovak Math. J. 2(77) (1952), 333–368; MR **15**, 815] imply the following: In the space of bounded continuous mappings of an n-dimensional metric space R into E^k ($k\leqq n$) the set of all mappings for which inverse images of points are at most $(n-k)$-dimensional is dense. It is stated that a special case of this result was conjectured by P. Aleksandrov and proved by V. Boltyanskiĭ.
J. Isbell (Seattle, Wash.).

Kowalsky, Hans-Joachim. Einbettung metrischer Räume. Arch. Math. **8** (1957), 336–339.

A star-space is a metric space obtained by taking a collection of closed unit intervals I_α, identifying the 0-points, and defining $\rho(x, y)=|x-y|$ if x,y are in the same I_α, $\rho(x,y)=|x|+|y|$ if x, y are in different I_α. Theorem: A topological space is metrizable if and only if it can be embedded in a countable cartesian product of star-spaces. Thus star-spaces play the same role in the characterization of metric spaces that closed intervals play in Urysohn's characterization of separable metric spaces.
E. Michael (Seattle, Wash.).

Ganea, Tudor. Stability of polyhedra and hyperspaces of compact sets. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 975–978, LXXXI. (Russian summary)

A compact metric space is called stable if there exists an $\varepsilon>0$ such that, for every ε-deformation f_t ($0\leqq t\leqq 1$) of X, f_1 is onto. Using an example of E. E. Floyd [Ann. of Math. (2) **64** (1956), 396–398; MR **18**, 141], the author proves Th. 1: There exists a stable, contractible, 3-dimensional, finite polyhedron. According to K. Borsuk [Fund. Math. **41** (1955), 168–202; MR **16**, 946], 2_c^X denotes the space of non-empty, compact subsets of X, metrized by the metric ρ_c, where $\rho_c(A, B)=\inf\{\delta\geqq 0|$ there exist continuous $f\colon A\to B$ and $g\colon B\to A$ which move points $\leqq\delta\}$. Using Th. 1, the author settles two problems of K. Borsuk [ibid.] by proving Th. 2: There exists a 3-dimensional, compact metric absolute retract M such that 2_c^M has infinitely many components and is not locally connected.
E. Michael (Seattle, Wash.).

Albrecht, F. A stable contractible 2-dimensional polyhedron. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1047–1049, LXXXVII. (Russian summary)

Using an example of K. Borsuk [Comment. Math. Helv. **8** (1935/6), 142–148], the author shows the existence of a stable [see preceding review for definition], contractible, 2-dimensional, finite polyhedron.
E. Michael.

Tumarkin, L. A. On Cantorian manifolds of an infinite number of dimensions. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 244–246. (Russian)

An n-dimensional Cantorian manifold is an n-dimensional compactum (compact metric space) which cannot be separated by any closed subset of dimension $\leqq n-2$. Hurewicz and Menger [Math. Ann. **100** (1928), 618–633] and Tumarkin [C. R. Acad. Sci. Paris **186** (1928), 420–422] proved that every n-dimensional compactum contains a subset which is an n-dimensional Cantorian manifold [see Hurewicz and Wallman, Dimension theory, Princeton, 1948, p. 94; MR **3**, 312].

The author defines an infinite-dimensional Cantorian manifold as an infinite-dimensional compactum which cannot be separated by any of its finite-dimensional closed subsets. In this sense, the Hilbert cube is an infinite-dimensional Cantorian manifold. The author gives an example of an infinite-dimensional compactum (imbedded in the Hilbert cube) which contains no infinite-dimensional Cantorian manifold.

In 1926, the author proposed the following problem: Does every infinite-dimensional compactum contain finite-dimensional compacta of arbitrarily large dimension? Van Heemert [Nederl. Akad. Wetensch., Proc. **49** (1946), 905–910; MR **8**, 397] showed that every infinite-dimensional compactum contains 1- and 2-dimensional compacta, but Tumarkin's problem has not yet been solved. In this paper, the author proves the following theorem: Let R be any infinite-dimensional compactum. Then either a) R contains finite-dimensional compacta of arbitrarily large dimension, or b) R contains an infinite-dimensional Cantorian manifold. The alternatives in the conclusion are not mutually exclusive.

The proof employs Urysohn diameters and a lemma proved by the author in another connection. Let F be a closed subset of a compactum. The Urysohn diameter $d_k(F)$ is the inf. of all $\varepsilon>0$ for which there exists a finite closed ε-covering of F of order $\leqq k$ (i.e., every $k+1$ sets of the covering have an empty intersection). The Urysohn diameters of a compactum form a non-increasing

sequence of nonnegative numbers and F is n-dimensional if and only if $d_n>0$ and $d_{n+1}=0$. F is infinite-dimensional only if $d_k>0$ for all k and $\lim d_k=0$. The lemma in question states that if A and B are closed subsets of a compactum and $\dim(A\cap B)\leq n-2$ (n a natural number), then at least one of the numbers d_nA, d_nB is equal to $d_n(A\cup B)$.

The proof of the theorem itself consists in constructing, on the assumption that a) does not hold, in the given infinite-dimensional compactum a Cantorian manifold of infinite dimension by transfinite induction. The author remarks that one could use instead Brouwer's reduction theorem [Alexandroff and Hopf, Topologie I, Springer, Berlin, 1935, p. 123]. *H. Komm* (Troy, N.Y.).

Keldych, Ludmila. Transformation of a monotone irreducible mapping into a monotone-interior mapping and a monotone-interior mapping of the cube onto the cube of higher dimension. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 472–475. (Russian)

Throughout, f denotes a monotone irreducible map of the continuum X onto the locally connected continuum Y. E denotes the set of all $y\in Y$ with $f^{-1}(y)$ a single point; E is then dense in Y. Two theorems, together with supporting lemmas, are announced; full proofs are to appear later. Th. 1: Suppose that Y is an n-manifold ($n\geq 3$) and that $U\cap E$ is connected for every region $U\subset Y$. Given $\varepsilon>0$, there exists an ε-displacement $\Phi: Y\rightarrow Y$ with Φf a monotone open map of X onto Y. Th. 2: Suppose that every region U of Y is not separated by any simple arc and also that $U\cap E$ is connected. There exists a monotone map $\Phi: Y\rightarrow Z$ with Φf a monotone open map onto Z and with $\dim Z\geq\dim Y-1$. The supporting lemmas are too long to be given. Among the consequences of the theorem are these: there is a monotone open map of the p-cell onto the q-cell for $3\leq p<q$; and there exists a monotone open map of the 3-cell onto a continuum, each region of which contains a copy of the Hilbert parallelotope. As is pointed out, Anderson has also announced results [e.g., Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 347–349; MR **17**, 1230] of a very similar nature.

E. E. Floyd (Charlottesville, Va.).

Hunter, R. P. Type of (n, k) adherence and indecomposability. Portugal. Math. **15** (1956), 115–122 (1957).

The paper is concerned with indecomposability properties of spaces. It contains misprints and obscurities which correspondence with the author leaves unresolved.

A. D. Wallace (New Orleans, La.).

Harrold, O. G., Jr.; Griffith, H. C.; and Posey, E. E. A characterization of tame curves in three-space. Trans. Amer. Math. Soc. **79** (1955), 12–34.

Les AA. énoncent deux conditions (P) et (Q) qui sont nécessaires et suffisantes pour qu'un cercle de R^3 soit non noué. (P) est une condition locale énoncée dans un article antérieur [Duke Math. J. **21** (1954), 615–621; MR **16**, 846]. (Q) exprime que le cercle J est bord d'un disque localement polyèdral, sauf sur J. Ample usage est fait des résultats récents de Graeub et Moïse. A noter également la démonstration d'un théorème dit des tores concentriques, donnant la condition pour que la portion d'espace comprise entre deux tores de même "âme" ait une adhérence homéomorphe au produit du tore par le segment [0—1].

R. Thom (Zbl **65** (1956), 167).

Albuquerque, L.; Dionisio, J.; and Farinha, J. Metric spaces and classical analysis: the method of fixed points. Gaz. Mat., Lisboa **16** (1955), no. 62, 1–6; **17** (1956), no. 63–64, 12–19. (Portuguese)

An expository article.

Algebraic Topology

Chang, Su-Cheng. On proper isomorphisms of (μ, Δ, γ)-systems. I. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 113–118.

A (μ, Δ, γ)-system, introduced by Chang and Whitehead [Quart. J. Math. Oxford Ser. (2) **2** (1951), 167–174; MR **13**, 374], consists of collections of groups $\{H^n\}$ and $\{H^n(2)\}$ and homomorphisms μ, Δ, and γ among certain of the groups. The motivating examples are obtained by using the cohomology groups, natural maps, and Steenrod squaring operations in finite simplicial complexes; hence invariants of isomorphism classes of (μ, Δ, γ)-systems obtained in this way are invariants of homotopy type of the complexes. In particular, it was shown in the paper mentioned above that, in certain cases, the Betti numbers, torsions, and some new invariants, called block invariants, determine the isomorphism class of the associated (μ, Δ, γ)-system. In the present paper, another list of cases is treated. It is necessary to introduce new invariants, called relative block invariants, and the main result is that these, together with the Betti numbers, torsions, and block invariants determine the isomorphism classes of the (μ, Δ, γ)-systems for the cases being examined.

T. R. Brahana (Athens, Ga.).

Nakaoka, Minoru. Cohomology of the p-fold cyclic products. Proc. Japan Acad. **31** (1955), 665–669.

Nakaoka, Minoru. Cohomology of the three-fold symmetric products of spheres. Proc. Japan Acad. **31** (1955), 670–672.

Nakaoka, Minoru. Cohomology theory of a complex with a transformation of prime period and its applications. J. Inst. Polytech. Osaka City Univ. Ser. A. **7** (1956), 51–102.

The author gives a complete exposition of the results reported in his papers listed above, with some additional results. In the earlier parts of this paper he studies the papers of R. Thom [Colloque de Topologie de Strasbourg, 1951, no. VII, Bibliothèque Nat. et Univ. de Strasbourg, 1952; MR **14**, 491] and W. T. Wu [ibid., no. IX; MR **14**, 491] concerning Steenrod's reduced powers. In the orbit space over a complex with a transformation of a prime period, the Smith-Richardson sequence and some basic homomorphisms in connection with the well-known cohomology operations are studied. Defining Thom's notion of regularity, two structure theorems (Th. 1 and 2 in Thom's note) are proved. Then the p-fold cyclic product of a complex is proved to be almost regular in every dimension (refer to Thom's Th. 3). The structure of the kernel of the homomorphism induced by the projection of the p-fold cartesian product on the cyclic product, is given by reduction formulas (refer to Thom's Th. 5) in terms of reduced powers. Then the reduced powers of Steenrod are characterized. The later parts of the paper are devoted to the calculation of the cohomology of cyclic products of special complexes. Let $C(A, q)$ be the q-primary component of the abelian group A, and let

$J\{A, r\}$ be the direct sum of r groups, each of which is isomorphic to A. Let us denote by $H^s(\mathfrak{Z}_p(S^n), Z)$ the s-dimensional integral cohomology group of the p-fold cyclic product $\mathfrak{Z}_p(S^n)$ of the n-sphere S^n. Then (i) $C(H^s(\mathfrak{Z}_p(S^n), Z), q)=0$ for any s and for $q\neq p, \infty$, (iia) $C(H^s(\mathfrak{Z}_p(S^n), Z), \infty)\cong Z$ if $s=0$ or pn with even $(p-1)n$, (iib) $C(H^s(\mathfrak{Z}_p(S^n), Z), \infty)\cong J\{Z, {}_pC_q/p\}$ if $s=nq$ with $1\leqq q\leqq p-1$, (iic) $C(H^s(\mathfrak{Z}_p(S^n), Z), \infty)=0$ for any other s, (iiia) $C(H^s(\mathfrak{Z}_p(S^n), Z), p)\cong Z_p$ if $(s-n)$ is odd and $3\leqq s-n\leqq(p-1)n$, (iiib) $\cong 0$ for any other s. Also the integral cohomology group of the p-fold cyclic product of a complex $Y^{n+1}(p^m)$, is calculated explicitly, where $Y^{n+1}(p^m)$ denotes the complex obtained by attaching an $(n+1)$-cell e^{n+1} to S^n by a map of degree p^m. Let us denote the 3-fold symmetric product of S^n by $S^n*S^n*S^n$; then $H^r(S^n*S^n*S^n, Z_2)\cong Z_2$ for $r=0, n, n+2\leqq r\leqq 2n, 2n+2\leqq r\leqq 3n$, and otherwise it is zero. $H^r(S^n*S^n*S^n, Z_3)$ is calculated completely. Generators of the groups and the integral homology of $S^n*S^n*S^n$ are also discussed. *H. Uehara.*

Dedecker, Paul. Cohomologie à coefficients non abéliens et espaces fibrés. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 1132–1146.

This paper is concerned with the problem of extending the structural group of a fibre bundle. This problem may be stated as follows: Let G be a topological group, N a closed normal subgroup of G, and $H=G/N$. Given a principal H-bundle $p':E'\to B$, find necessary and sufficient conditions that it be associated with a principal G-bundle $p:E\to B$ over the same base space B (it is assumed of course that G operates on H in the obvious way).

A solution to this problem has been found under the following additional hypotheses by J. Frenkel [C. R. Acad. Sci. Paris **240** (1955), 2368–2370; MR **16**, 1141] and the author [Colloque de topologie de Strasbourg, 1955, Inst. Math. Univ. Strasbourg, 1955; MR **16**, 1141; **19**, 302]: (i) N is contained in the center of G, (ii) N has a local cross section in G, and (iii) the base space B is paracompact.

In the present paper the author solves this problem without assuming hypothesis (i). In order to do this, he defines 2-dimensional cohomology of a simplicial complex with coefficients in a non-abelian group or of a space with coefficients in a sheaf of non-abelian groups. The cohomology classes thus defined actually depend on an exact sequence of coefficients rather than a single coefficient group or sheaf. *W. S. Massey.*

Dedecker, Paul. La structure algébrique de l'ensemble des classes d'espaces fibrés. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 270–290.

It is well-known that the classes of principal fiber spaces over B with group G correspond to the elements of $H^1(B, \mathfrak{G})$, where $\mathfrak{G}$ is the sheaf of germs of maps of B into G. The purpose of this paper is to imbed the set $H^1(B, \mathfrak{G})$ into an exact sequence; the author's cohomology with non-abelian coefficients is used [see the above review]. Any principal fiber space defines a fiber space E of groups over B, namely the associated fiber space with fiber G, the group G acting by inner automorphisms. Denote the sheaf of germs of sections of E by $\mathfrak{E}$. The author defines a set $\mathfrak{H}^1(B, \mathfrak{G})$ as the direct sum of the $H^1(B, \mathfrak{E})$ for all possible $\mathfrak{E}$; this set contains $H^1(B, \mathfrak{G})$. In this set a groupoid structure is introduced; this structure utilizes the notions of principal fiber space relative to a fiber space of groups, and of twisted fiber space (modification of the principal bundle of E by an element of $H^1(B, \mathfrak{E})$). The main result states that for an exact sequence of groups $e\to G^1\to G^2\to G^3\to e$, such that G^1 has a local cross section in G^2, there exists an exact sequence (of groupoids; the meaning of this is developed in the beginning of the paper) $e\to\mathfrak{H}^0(B, \mathfrak{G}^1)\to\cdots\to\mathfrak{H}^1(B, \mathfrak{G}^3)$, with a properly defined coboundary operator. It is stated that the sequence can be continued a few terms further.
H. Samelson (Ann Arbor, Mich.).

★ **Hirsch, Guy. Certaines opérations homologiques et la cohomologie des espaces fibrés.** Colloque de topologie algébrique, Louvain, 1956, pp. 167–190. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. français.

This paper is concerned with higher order cohomology operations. Such an operation associates to any n-tuple $u_1, \cdots, u_n$ $(n\geqq 1)$ of cohomology classes of the space X which satisfy certain natural conditions (these conditions are usually expressed by the vanishing of certain cohomology operations) a subset $T(u_1, \cdots, u_n)$ of some cohomology group of X. Moreover, it must satisfy the obvious naturality conditions.

Let $C^*(X)$, $Z^*(X)$, and $B^*(X)$ denote respectively the ring of cochains, cocycles, and coboundaries of the space X with coefficients in a field. The author assumes through this paper that there exist linear maps $\varkappa:H^*(X)\to Z^*(X)$ (where as usual $H^*=Z^*/B^*$) and $\sigma:B^*(X)\to C^*(X)$ such that the following two conditions are satisfied: (1) For any $u\in H^*(X)$, the cocycle $\varkappa(u)$ belongs to the cohomology class u. (2) For any coboundary v, $v=\delta\sigma(v)$, where "δ" denotes the coboundary operator. An important case where these conditions are satisfied is the case of the exterior differential forms on a Riemannian manifold; then one may define $\varkappa(u)$ to be the unique harmonic form in the class u.

The author gives an algebraic procedure for defining higher order cohomology operations wherever such homomorphisms $\varkappa$ and σ are defined. Moreover, due to the existence of these homomorphisms, the operations thus defined are unique, i.e., $T(u_1, \cdots, u_n)$ is a set consisting of one element. Of course the operations thus defined are not topological invariants, but depend on the choice of the homomorphisms $\varkappa$ and σ. The actual formulas and definitions used are very complicated.

One application of these methods is made: viz., to the problem of computing the cohomology of a fibre space. Here the author makes use of methods he has developed in earlier papers [see Bull. Soc. Math. Belg. **6** (1953), 79–96; MR **16**, 1142]. *W. S. Massey* (Providence, R.I.).

Vázquez, Roberto. Note on Steenrod squares in the spectral sequence of a fibre space. Bol. Soc. Mat. Mexicana (2) **2** (1957), 1–8. (Spanish)

The author applies his explicit definition of the cup-i product of singular cubical cochains [same Bol. **11** (1954), 9–32; MR **17**, 654] to fibre spaces E, where the filtration makes computations easier (because its cochains vanish on cubes of lower filtration). Defining $\phi_i(f)$ by $f\cup_i f+f\cup_{i+1}df$ (where f is a cochain and d the differential operator), the author shows that for $f\in C_r^{*p,q}$ [notations of Serre's spectral sequence, Ann. of Math. (2) **54** (1951), 425–505; MR **13**, 574], ϕ_i defines an element of $C_w^{*u,v}$ and (for $r\geqq 2$) of $E_w^{*u,v}$, for suitable values of u, v, w (depending on i, p, q, r). Hence, ϕ_i induces homomorphisms: $\bar\phi_i^1: E_r^{*p,q}\to E_s^{*2p-i,2q}$ for $i\leqq p-r+2$, $2r-2\leqq s\leqq 2r-1$; $\bar\phi_i^2$: $E_r^{*p,q}\to E_s^{*2p-i,2q}$ for $p-r+2\leqq i\leqq p$, $r-(i-p)\leqq s\leqq 2r-1$; $\bar\phi_i^3$: $E_r^{*p,q}\to E_s^{*p,2q-(i-p)}$ for $p\leqq i\leqq$

$p+r-1$, $r \leq s \leq 2r-(i-p)-1$; $\bar{\phi}_i^4$: $E_r^{*p,q} \to E_r^{*p,2q-(i-p)}$ for $p+r-1 \leq i$.

In E_∞, one has $\bar{\phi}_i^2$: $E_\infty^{*p,q} \to E_\infty^{*2p-i,2q}$ for $i \leq p$, and $\bar{\phi}_i^3$: $E_\infty^{*p,q} \to E_\infty^{*p,2q-(i-p)}$ for $i \geq p$, which are induced by Sq_i in the term $D^{*p,q}$ of the composition series of $H^*(E; Z_2)$.

Because of the isomorphism $E_2^{*p,q} \approx H^p(B; H^q(F; Z_2))$, $\bar{\phi}_i^1$ (for $i \leq p$) and $\bar{\phi}_i^4$ (for $i > p$) can be interpreted as being homomorphisms of $H^p(B; H^q(F; Z_2))$ in $H^{2p-i}(B; H^{2q}(F; Z_2))$ and in $H^p(B; H^{2q-(i-p)}(F; Z_2))$, respectively; they correspond respectively to the homomorphism Sq_i in the base space B, using the cup product for the coefficients (belonging to $H^*(F; Z_2)$, and to the homomorphism induced in $H^*(B)$ by the homomorphism Sq_i applied to the coefficients belonging to $H^*(F; Z_2)$.

G. Hirsch (Brussels).

James, I. M. Note on cup-products. Proc. Amer. Math. Soc. **8** (1957), 374–383.

Suppose that n and q are integers such that $q>1$ and $n>q+1$, and let K be a cell complex obtained by attaching an n-cell, e^n, to a q-sphere, S^q, by a map of the boundary of e^n into S^q representing an element $\alpha \in \pi_{n-1}(S^q)$ and then attaching an $(n+q)$-cell, e^{n+q}, to the complex thus obtained by some map of the boundary of e^{n+q}. Let x, y, and z denote generators of the integral cohomology groups of K in dimensions n, q, and $n+q$ respectively (these cohomology groups are all infinite cyclic). Let m be the unique integer such that $xy=mz$ in the cohomology ring of K. Then the integer m and the homotopy class α are invariants of the homotopy type of K. The main theorem of the present paper asserts that there exists such a cell complex $K=S^q \cup e^n \cup e^{n+q}$ of type (m, α) if and only if there exists an element $\beta \in \pi_{n+q-2}(S^{n-1})$ such that $m[\alpha, \iota_q] = \alpha \circ \beta$ (in this formula, ι_q denotes a generator of $\pi^q(S^q)$, square brackets denote the Whitehead product, and $\alpha \circ \beta$ denotes the composition of the homotopy classes α and β).

In addition to proving this main theorem, the author shows two different directions in which it can be applied. Firstly, he gives examples to show that known theorems about Whitehead products in homotopy groups of spheres can be applied to deduce the existence or nonexistence of complexes $K=S^q \cup e^n \cup e^{n+q}$ of type (m, α) for various m and α. Secondly, Adem's relations on iterated Steenrod reduced powers [Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 720–726; **39** (1953), 636–638; MR **14**, 306; **15**, 53] imply that the cohomology ring of a space must satisfy certain conditions. By using these conditions for $K=S^q \cup e^n \cup e^{n+q}$ when $n=2q$ and q is even, the author is able to deduce certain conditions on Whitehead products in homotopy groups of spheres. *W. S. Massey* (Providence, R.I.).

Livesay, G. R. Concerning real valued maps of the n-sphere. Proc. Amer. Math. Soc. **8** (1957), 989–991.

The author proves the following theorem: Let S^n be the unit sphere in the euclidean $(n+1)$-space and let p be the projection of S^n into the projective n-space P^n of antipodal pairs on S^n. If f is a continuous real-valued function on S^n, $0 \leq d \leq 2$ and $X_d = \{x \in S^n \mid$ there exists $y \in S^n$ with $\|x-y\|=d$, $f(x)=f(y)\}$, then pX_d carries a non-trivial mod 2 Čech $(n-1)$-cycle of P^n. *C.T. Yang.*

★ **Whitehead, George W. Homotopy theory. Compiled by Robert J. Aumann.** Mathematics Department, Massachusetts Institute of Technology, Cambridge, Mass., 1953. i+168 pp. $3.50.

Huebsch, William. On the covering homotopy theorem. Ann. of Math. (2) **61**, 555–563 (1955).

Das "covering homotopy theorem" (C.H.T.) von Hurewicz-Steenrod betrifft die Existenz einer covering Homotopie für eine gegebene Homotopie $B \times I \to B^*$ der Basis B eines gefaserten Raumes X in die Basis B^* eines gefaserten Raumes X^*. In dem Buch von Steenrod [The topology of fibre bundles, Princeton, 1951; MR **12**, 522] wird das C.H.T. für Faserbündel bewiesen unter der Voraussetzung, daß B ein C_σ-Raum ist, d.h. ein normaler, lokal-kompakter Raum mit der Eigenschaft, daß es zu jeder Überdeckung mit offenen Mengen eine abzählbare Teilüberdeckung gibt. Verf. gibt einen vollständigen Beweis für das C.H.T. und zeigt, daß es genügt, B als parakompakt vorauszusetzen. Verf. führt seine Überlegungen durch für gefaserte Räume im Sinne von Hu [Proc. Amer. Math. Soc. **1** (1950), 756–762; MR **12**, 435].

F. Hirzebruch (Zbl **65** (1956), 388).

Huebsch, William. Covering homotopy. Duke Math. J. **23** (1956), 281–291.

Verf. setzt seine Untersuchungen über das covering homotopy theorem fort. Die früheren Ergebnisse werden verallgemeinert. Covering homotopy-Sätze werden für beliebige Tripel (E, B, p), wo p eine stetige Abbildung von E in B ist, formuliert. Unter gewissen Bedingungen impliziert eine lokale covering homotopy-Eigenschaft bereits das globale covering homotopy-Theorem. In einer bei der Korrektur hinzugefügten Bemerkung werden Zusammenhänge mit einer Arbeit von Hurewicz diskutiert [vgl. Hurewicz, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 956–961; MR **17**, 519]. *F. Hirzebruch* (Zbl **71** (1957), 170).

Dugundji, J. Remark on homotopy inverses. Portugal. Math. **14** (1955), 39–41.

Let C be the category consisting of a set, A, of spaces, and the set of homotopy classes of maps $X \to Y$, $X, Y \in A$. A covariant functor, G, from the category C to the category of groups and homomorphisms is called an h-system on A. If G_1, G_2 are two h-systems on A, and $Y \in A$, a homomorphism $\eta: G_1(Y) \to G_2(Y)$ is called natural if $\eta G_1(\alpha) = G_2(\alpha)\eta$ for all homotopy classes α: $Y \to Y$. (The author uses a different notation here.) It is shown that if η is an isomorphism and if, for some $X \in A$, there exist maps $f: X \to Y$, $g: Y \to X$ with gf homotopic to the identity (i.e., if Y dominates X), then $G_1(X)$ is isomorphic with $G_2(X)$. This result is used to prove that, in any space dominated by a CW-polytope, the Čech homology groups based on all coverings and the singular homology groups are isomorphic in each dimension. The reviewer remarks that this also follows from the fact that a space dominated by a CW-polytope has the homotopy type of a CW-polytope. *P. Hilton* (Zbl **65** (1956), 385).

★ **Uehara, Hiroshi; and Massey, W. S. The Jacobi identity for Whitehead products.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 361–377. Princeton University Press, Princeton, N. J., 1957. $7.50.

Let X be a topological space with base-point, and let $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, $\gamma \in \pi_r(X)$, $p, q, r \geq 2$. Then the well-known 'Jacobi identity' asserts that

$$(-1)^{p(r+1)}[\alpha, [\beta, \gamma]] + (-1)^{q(p+1)}[\beta, [\gamma, \alpha]] + (-1)^{r(q+1)}[\gamma, [\alpha, \beta]] = 0.$$

The authors prove the Jacobi identity for $p, q, r > 2$ by

first analysing the algebraic structure of $\pi_{\rho-2}(X)$, where $X=S^p\vee S^q\vee S^r$, $\rho=p+q+r$ (Th. I) and then proving that if a relation of the form $l[\alpha,[\beta,\gamma]]+m[\beta,[\gamma,\alpha]]+n[\gamma,[\alpha,\beta]]=0$ holds in $\pi_{\rho-2}(X)$ for this special X, where α, β, γ are the homotopy classes of S^p, S^q, S^r in X, then $(-1)^{p(r+1)}l=(-1)^{q(p+1)}m=(-1)^{r(q+1)}n$ (Th. II). Th. I was proved by the reviewer [J. London Math. Soc. **30** (1955), 154–172; MR **16**, 847], and Th. II is very closely related to a result of Samelson [Amer. J. Math. **75** (1953), 744–752; MR **15**, 731]. The authors' proof of Th. II is of considerable interest, since it uses Massey's second order cohomology operation, the triple product $\langle u, v, w\rangle$ of $u\in H^p(K)$, $v\in H^q(K)$, $w\in H^r(K)$, defined if $uv=0$, $vw=0$, $uw=0$. Basic properties (including naturality) of this triple product are proved, and the product $\langle u, v, w\rangle$ is computed, where X, α, β, γ are as above, K is obtained from X by attaching a $(\rho-1)$-cell by a map in the class $m[\alpha,[\beta,\gamma]]$, and u, v, w are the cohomology classes of the spheres S^p, S^q, S^r in X. Th. II is readily deduced from this computation. *P. J. Hilton.*

Toda, Hirosi; Saito, Yoshihiro; and Yokota, Ichiro. Note on the generator of $\pi_7(\mathrm{SO}(n))$. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 227–230.

Let $\sigma:S^7\to\mathrm{SO}(8)$ and $\rho:S^7\to\mathrm{SO}(7)$ be defined by $\sigma(x)(y)=xy$ and $\rho(x)(y)=xyx^{-1}$ for $x, y\in S^7$ (=unit Cayley numbers). Let $\sigma_n\in\pi_7(\mathrm{SO}(n))$ for $n\geqq 8$ and $\rho_n\in\pi_7(\mathrm{SO}(n))$ for $n\geqq 7$ denote the corresponding homotopy classes. Then the authors prove that $\pi_7(\mathrm{SO}(7))$ is an infinite cyclic group generated by ρ_7, $\pi_7(\mathrm{SO}(8))=Z+Z$, generated by σ_8 and ρ_8, and $\pi_7(\mathrm{SO}(n))$ for $n\geqq 9$ is an infinite cyclic group generated by σ_n. These groups had been determined by Serre [C. R. Acad. Sci. Paris **236** (1953), 2475–2477; MR **14**, 1110], without details. *F. P. Peterson.*

Tamura, Itiro. On Pontrjagin classes and homotopy types of manifolds. J. Math. Soc. Japan **9** (1957), 250–262.

The author gives examples of differentiable manifolds which are of the same homotopy type but which have different Pontrjagin classes, and consequently are not differentiably homeomorphic. Thus either the Pontrjagin classes are not topological invariants or the Hurewicz conjecture that manifolds of the same homotopy type are homeomorphic is false.

The author uses results of Dold [Math. Z. **62** (1955), 111–136; MR **17**, 519] on the homotopy type of sphere bundles over spheres, to show that the manifolds are of the same homotopy type, and most of the paper is devoted to a direct computation of the Pontrjagin classes.

As the author points out to me in a letter, the diagram on p. 255 can be made commutative by an appropriate choice of the maps. This fact is used later on. The left square in the diagram on p. 253 is only commutative up to sign, as was shown by J. H. C. Whitehead [Ann. of Math. (2) **58** (1953), 418–428; MR **15**, 642] but this does not affect the argument. Theorem 5.4 is stated in a misleading way. These groups are some, not all, of the nontrivial cohomology groups of the spaces in question. In formula (2.1) and throughout the rest of the paper, $t=1$, as is shown by Toda, Saito and Yokota [article reviewed above].

The author remarks that using one of his results and methods analogous to Milnor's [Ann. of Math. **64** (1956), 399–405; MR **18**, 498], one can construct topological 15-spheres with inequivalent differentiable structures.

N. Stein (New Haven, Conn.).

Curtis, M. L.; and Fort, M. K., Jr. Certain subgroups of the homotopy groups. Michigan Math. J. **4** (1957), 167–172.

Let X be an arcwise connected separable metric space and denote by $F_n(X, x_0)$, for a fixed $x_0\in X$, the set of all continuous maps of $(S^n, y_0)\to(X, x_0)$, where S^n is the unit n-sphere in E^{n+1} with center at the origin and $y_0=(1, 0, \cdots, 0)$. Using two auxiliary unit spheres and orientation-preserving affine transformations, the authors define an operation on pairs $f, g\in F_n(X, x_0)$. If $\pi_n(X, x_0)$ is the set of homotopy classes of $F_n(X, x_0)$ obtained from homotopies with fixed base point, the operation induced on π_n by the operation defined on F_n converts π_n into the nth homotopy group of X with base point x_0. Call a subset S of F_n a G-class if $\psi(S)$ is a subgroup of π_n, where $\psi:F_n\to\pi_n$ is the natural map. The paper investigates G-classes S which are closed under both the operation on F_n and composition with the reflection of S^n about the plane $z_{n+1}=0$, and are further restricted by imposing either a condition on each inverse image of a point (Type I) or a condition on all but a finite number of the inverse images of points (Type II), for each $f\in S$. To obtain an example of a G-class of Type II, define $f:Y\to W$ to be k-light if $\dim f^{-1}(w)\leqq k$ for all but a finite number of $w\in W$. The subset of F_n consisting of all k-light maps is a G-class of Type II. Call the corresponding subgroup $D_n^k(X, x_0)$ of π_n, the k-light subgroup of π_n. It is shown that $D_1^0=\pi_1$ and that if X is k-dimensional, $D_n^m=0$ for $n>m+k$. If X is a k-manifold, $D_n^m=\pi_n$ for $n\leqq m+k$. The last result does not hold if X is only an algebraic variety.

For a suitably restricted homology theory, the property of having a finitely generated k-dimensional homology group for the inverse image of each point defines a G-class of Type I, the class of k-monotone maps. The corresponding subgroup of π_n, the k-monotone subgroup, coincides with π_n if X is a finite polyhedron.

The k-light groups depend on the base point x_0; the k-monotone groups do not. It is pointed out that the subgroups corresponding to G-classes are not homotopy invariants. *H. Komm* (Troy, N.Y.).

Adams, J. F. An example in homotopy theory. Proc. Cambridge Philos. Soc. **53** (1957), 922–923.

The example answers negatively the question (due to J. H. C. Whitehead): are two CW-complexes of the same homotopy type if they are of the same n-type for each n? A simple induction argument shows, however, that the answer is "yes" when all homotopy groups of the complexes are finite. *J. A. Zilber* (Providence, R.I.).

Mennicke, J. Über Heegaarddiagramme vom Geschlecht zwei mit endlicher Fundamentalgruppe. Arch. Math. **8** (1957), 192–198.

Using Nielsen's generators for the group of automorphisms [Danske Vid. Selsk. Math.-Fys. Medd. **15** (1937), no. 1] of the fundamental group of the surface of genus 2 [Coxeter and Moser, Generators and relations for discrete groups, Springer, Berlin, 1957, p. 61; MR **19**, 527], the author obtains defining relations for the fundamental groups of certain three-dimensional manifolds. In each case, the group is found to be the direct product of a cyclic group and a binary polyhedral group. The author seems to have overlooked the fact that the relations

$$A_1^m=A_2^n=(A_1A_2)^2$$

suffice to define the binary polyhedral group of order

$8mn/k$, where $k=2m+2n-mn>0$ [Coxeter and Moser, ibid., p. 69]; there is no need to insert the extra relation $A_1^{2m}=1$. *H. S. M. Coxeter* (Toronto, Ont.).

Papakyriakopoulos, C. D. On the ends of knot groups. Ann. of Math. (2) **62** (1955), 293–299.

(*) Ist der Außenraum eines Knotens in der S^3 asphärisch und besitzt seine Fundamentalgruppe G zwei Enden, so ist G unendlich zyklisch. Verf. folgert dies aus den beiden folgenden Sätzen: Sei M eine asphärische berandete dreidimensionale Mannigfaltigkeit mit unendlicher Fundamentalgruppe G und der Eigenschaft, daß eine Kurve einer Randfläche F von M nur dann im M homotop 0 ist, wenn sie schon auf F homotop 0 ist; dann hat G ein Ende. Sei M eine asphärische dreidimensionale Mannigfaltigkeit, die von einem Torus T berandet wird; es gebe auf T eine Kurve, die nicht auf T, wohl aber in M homotop 0 ist; dann ist M endlich und orientierbar und die Fundamentalgruppe von M ist unendlich zyklisch. Bemerkung: (*) kann folgendermaßen verschärft werden: Besitzt die Fundamentalgruppe eines asphärischen Raumes zwei Enden, so ist sie unendlich zyklisch. Es folgt dies daraus, daß die Fundamentalgruppe eines asphärischen Raumes kein Element endlicher Ordnung enthält [P. A. Smith, Ann. of Math. (2) **39** (1938), 127–164] und daß eine Gruppe mit unendlich zyklischer Untergruppe von endlichem Index (d.h. eine Gruppe mit zwei Enden) ohne Elemente endlicher Ordnung unendlich zyklisch ist.

E. Specker (Zbl **67** (1957), 158).

Kyle, R. H. Embeddings of Möbius bands in 3-dimensional space. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 131–136.

Verf. betrachtet semilineare Einbettungen des Möbiusbandes und ihre Äquivalenz bezüglich semilinearer Isotopien der 3-Sphäre. Es wird gezeigt: (1) Zwei Einbettungen sind äquivalent, wenn es ihre Mittellinien sind und wenn sie dieselbe Verdrillung besitzen. (2) Der Einbettungstyp ist durch den Typ des Randes bestimmt, wenn dieser verknotet ist. Zu unverknotetem Rand gibt es zwei Typen von Einbettungen. Bei diesen ist die Mittellinie unverknotet und die Verdrillung ± 1. Für die Einbettungen des Kreisringes überträgt sich (1) ohne weiteres, während an Stelle von (2) die Bermerkung tritt, daß die Ränder eines Kreisringes bei gleichsinniger Orientierung seine Verknotung und Verdrillung bestimmen.

Horst Schubert (Zbl **66** (1956), 171).

See also: General Topology: Ganea; Albrecht; Harrold, Griffith and Posey. **Differential Geometry:** Segre. **Programming, Resource Allocation, Games:** Berge.

GEOMETRY

See: Probability: Tukey.

Geometries, Euclidean and Other

Pikus, D. L. On an axiom of triangle congruence in weakened form. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 359–362. (Russian)

The author constructs a geometry resembling a model of a non-pythagorean geometry given by Hilbert [Die Grundlagen der Geometrie, 5th ed., Teubner, Leipzig, 1922, appendix II; see also W. Rosemann, Math. Ann. **90** (1923), 108–128]: points and lines are the points and lines of the euclidean plane; congruence is defined by the group of congruence transformations $x'=e^{\lambda(\alpha)}(x\cos\alpha-y\sin\alpha)+a$, $y'=e^{\lambda(\alpha)}(x\sin\alpha+y\sin\alpha)+b$, where $\lambda(\alpha)$ is a (discontinuous) solution of the functional equation $\lambda(\alpha+\beta)=\lambda(\alpha)+\lambda(\beta)$ satisfying $\lambda(\pi)=0$, and hence $\lambda(r\pi)=0$ for rational r. This geometry satisfies Hilbert's axioms I. 1–3 (incidence), II. 1–3 (order), III. 1–4 (congruence), IV (the parallel axiom), V. 1–2 (the axioms of Archimedes and linear completeness), and the weak congruence axiom III.5* (equally oriented triangles are congruent if two pairs of sides and the enclosed angles are congruent), but not the strong axiom III.5 (asserting the congruence of oppositely oriented triangles as well). The author concludes that Hilbert's statement that III.5 is a consequence of I.1–3, II.1–3, III.1–4, III.5*, IV, V.1–2 is erroneous; this appears to be a misunderstanding, as in Hilbert's statement V.2 is not the axiom of completeness but the neighbourhood axiom: to every line segment AB there exists a triangle whose interior does not contain any line segment congruent to AB. This axiom fails to hold in the author's geometry. The author does not seem to be aware of the results of A. Schmidt [Math. Ann. **109** (1934), 538–571] and P. Bernays [Courant Anniversary Volume, Interscience, New York, 1948, pp. 29–44; MR **9**, 244] concerning the relation between the axioms III.5 and III.5*. The author's model was used, in a different context, by G. Pickert [Math. Ann. **120** (1949), 492–501; MR **10**, 571], and in a paper just published [Arch. Math. **8** (1958), 477–480] H. Lenz notes its significance for establishing the independence of the strong axiom III. 5.

F. A. Behrend (Melbourne).

Thébault, Victor. Recreational geometry — the triangle. Scripta Math. **22** (1956), 97–105.

Conclusion of the article reviewed in MR **18**, 411.

★**Müller, Emil; und Kruppa, Erwin. Lehrbuch der darstellenden Geometrie. 5te, ergänzte Aufl.** Springer-Verlag, Vienna, 1948. ix+404 pp. $4.80.

This book has been well known since 1908. The present edition differs from the 4th edition of 1936 chiefly in the addition of a chapter on photogrammetry.

Convex Domains, Integral Geometry

Pucci, Carlo. Sulla inscrivibilità di un ottaedro regolare in un insieme convesso limitato dello spazio ordinario. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 61–65.

Broch, E. K. Some remarks concerning the plane lattice and close-packing of equal circles. Avh. Norske. Vid. Akad. Oslo. I. **1956**, no. 3, 7 pp.

In this essay, the author describes the closest lattice packing of circles in the plane, and explains why no "twinning" of the lattice is possible. He promises to explain in a subsequent paper how simple and multiple twinformations can occur in the closest lattice packings of

spheres in three dimensional space, and how this leads to an infinite number of different close-packed structures.
C. A. Rogers (Birmingham).

Danzer, L. Über ein Problem aus der kombinatorischen Geometrie. Arch. Math. 8 (1957), 347–351.

It is shown that if $\{P_i\}$ is a set of at least 5 points in the Euclidean plane such that $|P_iP_j|\geqq 1$ for $i\neq j$, and such that every subset of 5 points can be covered by a strip of width 1, then $\{P_i\}$ can be covered by such a strip. This result is also formulated in terms of disjoint unit circles. In the above, the number 5 cannot be reduced to 4, but it is conjectured that the result remains valid if $|P_iP_j|\geqq 1$ is replaced by $|P_iP_j|\geqq 2((5-2\cdot 5^{\frac{1}{2}})/5)^{\frac{1}{2}}$. A number of related unsolved problems are mentioned. *L. Moser.*

Bieri, H. Untersuchungen über rotationssymmetrische Kegelstümpfe. Collect. Math. 8 (1955–1956), 171–185.

The problem to determine the set of all triples of real numbers (V, F, M) such that there exists a convex body in ordinary space for which V is the volume, F the surface area, and M the integral of the mean curvature, is not yet solved completely. An inequality giving the minimum of V for given F and M satisfying $2M^2<\pi^3F$ is unknown. For convex bodies of revolution, the corresponding inequality has been found by H. Hadwiger [Portugal. Math. 7 (1948), 73–85; MR 10, 471]. However under this restriction, new difficult problems of this type arise. In order to obtain some preliminary information about them, the author restricts the class of bodies much further, viz., to the truncated cones (cylinders and cones included) of revolution. In terms of the parameters $x=4\pi FM^{-2}$, $z=36\pi V^2F^{-3}$, the problem consists in determining a certain region in the xz-plane. Explicit expressions for the greater part of the boundary of the region are found in an elementary, but troublesome way, and the bearing of the results on the problem for general bodies of revolution is discussed. *W. Fenchel.*

Baebler, F. Zum isoperimetrischen Problem. Arch. Math. 8 (1957), 52–65.

The author gives a comparatively simple proof [related to those of A. Dinghas, Math. Nachr. 2 (1949), 107–113; MR 11, 386; and H. Hadwiger, Portugal. Math. 8 (1949), 89–93; MR 12, 353] of the isoperimetric inequality between the volume (i.e. the Lebesgue measure) and the surface area (in the sense of Minkowski) of an arbitrary closed point set in Euclidean n-space. Equality holds, not only for the solid spheres, but also for certain other closed sets. The weakest restriction of the class of closed sets known to ensure that equality holds only for spheres, requires that the sets B of the class have the following property: Every solid sphere with center in B intersects B in a set of positive n-dimensional measure. This result, due to E. Schmidt [cf., e.g., Math. Nachr. 1 (1948), 81–157; 2 (1949), 171–244; MR 10, 471; 11, 534], is proved in a new way. *W. Fenchel* (Copenhagen).

Lenz, Hanfried. Eine Kennzeichnung des Ellipsoids. Arch. Math. 8 (1957), 209–211.

Let E be a non-degenerate convex body in real Euclidean n-space, symmetric with respect to the origin O. A_i, $1\leqq i\leqq n$, are boundary points of E, while A_i' is the reflection of A_i with respect to O. V is defined to be the minimum of the volumes of the parallelotopes bounded by hyperplanes supporting E at A_i, A_i', $1\leqq i\leqq n$, while v is the maximum of the volumes of the convex hulls of the point sets A_i, A_i', $1\leqq i\leqq n$. If either E is supported by a unique hyperplane at each boundary point or each of its supporting hyperplanes contains exactly one point of E, the author shows $V/v\leqq n!$. If these two numbers are equal, he deduces for $n>2$ that E is an ellipsoid. In course of the proof it follows that, for every E, at least two sets of points A_i, $1\leqq i\leqq n$, exist so that OA_i is parallel to the supporting hyperplanes of E at A_j, $j\neq i$. *D. Derry.*

Hadwiger, H. Über Treffanzahlen bei translationsgleichen Eikörpern. Arch. Math. 8 (1957), 212–213.

A is a non-degenerate convex body in real k-dimensional Euclidean space. S is a set consisting of A and translations of A every element of which contains at least one boundary point of A, but for which no two different elements contain a common interior point. Where n is the number of elements in S, the author proves that $n\leqq 3^k$, and points out that $n=3^k$ if A is a parallelotope, and the members of S belong to the lattice generated by A. The following problem is proposed. If S contains the largest possible number of elements, find the minimum value of n for arbitrary A. *D. Derry.*

Differential Geometry

Wintner, Aurel. On the curvatures of a surface. Amer. J. Math. 78 (1956), 117–136.

This is one of a series of papers by the author — some written jointly with P. Hartman — on the basic equations of surface theory, with special regard to the differentiability assumptions. A piece of surface S defined by a vector function $X(u, v)$ of local parameters (u, v) is of class C^n if, for some local coordinates about any point, the function $X(u, v)$ is of class C^n. Several topics are discussed.

1. Although the mean curvature $H(u, v)$ involves the second derivatives of X, the author introduces a generalized notion of mean curvature H such that some surfaces S in C^1 but not in C^2 (as shown by an example) possess such a continuous generalized H. Denoting by $\times$ the vector product, and by N the unit normal, a continuous function $H(u, v)$ is called the generalized mean curvature for $S\in C^1$ (not every $S\in C^1$ has one) if for any simple piecewise smooth Jordan curve J bounding a domain B we have

$$\int_J N\times dX=-2\iint_B HNdA,$$

where dA is the surface element of area. In connection with this definition some problems are formulated.

2. An analogous generalized formulation of the Codazzi-Mainardi equations for a C^1 representation of a surface in C^2 is given. It is shown that such representations arise naturally in the theory of ruled surfaces.

3. Suppose S has a C^2 representation $z=z(x, y)$ over a domain B in the (x, y) plane with area $|B|$ and boundary curve J, having length $|J|$. If $|I|$ is the length of the spherical image mapping of J, then simple proofs of the following inequalities [see also E. Heinz, Math. Ann. 129 (1955), 451–454; MR 17, 189] are given: $|J|/|B|\geqq \min_{B+J}2|H|$, $|I|/|B|\geqq\min_{B+J}2|K|$. Here K is the Gauss curvature.

4. Using some results of Schilt [Compositio Math. 5 (1937), 239–283], the author exhibits a piece of a 2-dimensional Riemannian manifold with positive Gauss curvature which becomes infinite at a point P, such that no neighborhood of P admits a C^1 isometric embedding as a convex surface in 3-space. *L. Nirenberg.*

Zezula, Jaromír. Metrische Charakterisation einer windschiefen Regelfläche mit uneigentlicher Fleknodalkurve. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 6 (1956), 205–207. (Czech. Russian and German summaries)

Die uneigentliche Kurve einer windschiefen Regelfläche ist ihre Fleknodalkurve dann und nur dann, wenn das Produkt der Halbachsen ihres Oskulationshyperboloides konstant ist. *Zusammenfassung des Autors.*

★ **Лопшиц, А. М. [Lopšic, A. M.] Вычисление площадей ориентированных фигур. [Calculation of areas of oriented figures.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 59 pp. 0.90 rubles.

A popular exposition leading up to the theory of planimeters.

Helfenstein, H. G. Geodesic groups of minimal surfaces. Canad. J. Math. 10 (1958), 89–96.

All pairs of minimal surfaces which can be geodesically mapped on each other are either 1) similar Bonnet surfaces of each other, 2) both isometric to a plane (Poisson surface), or 3) both Scherk surfaces. In case 1) the mappings are combinations of trivial transformations with the self-isometries of a minimal surface. In case 2) the mappings are generated by the projectivities of a complex plane followed by isometries and similarities. In case 3) they are those of the geodesic group of a single Scherk surface together with trivial transformations. The geodesic mappings of a Scherk surface onto itself form a non-abelian, intransitive, mixed discrete-continuous group G. Its finite part is a non-abelian group F of order 16; its 21 proper subgroups, all abelian, are enumerated. *D. J. Struik* (Cambridge, Mass.).

Rembs, Eduard. Randvorgaben bei infinitesimaler Verbiegung einfach zusammenhängender konvexer Flächen. Monatsh. Math. 60 (1956), 57–65.

The author discusses the uniqueness and existence of infinitesimal isometric deformations of simply connected pieces of convex surfaces, assuming that the (infinitesimal) variation of one of the following quantities is prescribed: (1) the curvature of the boundary curve; (2) the geodetical torsion of the boundary strip; (3) the angle formed by the principal normal of the boundary curve and the normal of the boundary strip. These problems are reduced to boundary value problems for elliptic systems of two linear partial differential equations [cf. W. Haack, Math. Nachr. 8 (1952), 123–132; MR 14, 986]. Let $x(u, v)$ and $y(u, v)$ denote the position vectors of the given surface and of the "Drehriss" belonging to an infinitesimal deformation, respectively. Possibly restricting the class of surfaces, it is assumed that the parametric representation is isothermic-conjugate, i.e., such that the coefficients of the second fundamental form satisfy $L=N$, $M=0$. Then $y_u=\alpha x_u+\beta x_v$, $y_v=\beta x_u-\alpha x_v$, and the Codazzi equations yield a linear system with α and β as unknown functions. In terms of α and β the boundary conditions formulated above are all of characteristic (index) 2 in the sense of Haack. [In the cases (2) and (3) this is only shown for spherical caps. However, this restriction has been removed by the author in the paper reviewed below.] From known results concerning such boundary value problems, uniqueness theorems (and thus rigidity theorems) are inferred, which are new in cases (2) and (3) and more general than the one previously obtained by the author [Abh. Math. Sem. Univ. Hamburg 20 (1956), 178–185; MR 18, 760] in case (1). It follows further that solutions exist if and only if in each case the prescribed variation satisfies three integral conditions depending on solutions of the adjoint system. These conditions are given explicitly for a spherical cap. *W. Fenchel* (Copenhagen).

Rembs, Eduard. Bemerkung zu meiner Arbeit "Randvorgaben bei infinitesimaler Verbiegung konvexer Flächen". Monatsh. Math. 60 (1956), 212–213.

In 3-space, a convex surface with a boundary will be rigid if the variation δa of the geodetic torsion is held to zero on the boundary, and if the surface be infinitesimally bent, the consequent (non-zero) variation δa must satisfy three conditions of an integral type. These results are analogous to those obtained in previous papers [cf. the paper reviewed above; and Arch. Math. 6 (1955), 55–58; MR 16, 1050] concerned with variations in the curvature of the boundary. *A. Douglis* (New York, N.Y.).

Rembs, E. Ein Biegungsproblem mit negativer Charakteristik. Monatsh. Math. 60 (1956), 333–336.

It is shown that every simply connected piece of a convex surface admits infinitely many infinitesimal (isometric) deformations such that each point of the boundary curve is moved in a direction normal to the curve (actually in the direction of the binormal). The deformation is uniquely determined (apart from a constant factor) by the requirement that a prescribed interior point of the surface is moved in the direction of the normal of the surface. These deformations are infinitesimal rigid motions if and only if the boundary is a plane curve. A method due to T. Minagawa and T. Rado [Osaka Math. J. 4 (1952), 241–285; MR 14, 794] is used to reduce the problem to a boundary value problem of characteristic (index) -1 for an elliptic system of two linear partial differential equations [cf. W. Haack, Math. Nachr. 8 (1952), 123–132; MR 14, 986]. *W. Fenchel* (Copenhagen).

Čech, Eduard. Zur projektiven Differentialgeometrie. Schr. Forschungsinst. Math. 1 (1957), 138–142.

This is a brief sketch of results, due to the author and others, and concerning mostly the projective differential and X_n-differential contact invariants of manifolds in an n-dimensional space X_n; here, X_n-invariant means invariant under an arbitrary (local) invertible analytic transformation. We mention what seems to be the main result. Let V_r, V_r' be two r-dimensional submanifolds of X_n ($r\leqq n$), and let $x^0(x_i^0)$ be a point of $V_r\cap V_r'$. An analytic 1-1 correspondence $V_r\leftrightarrow V_r'$, defined in the neighborhood of x^0, and such that

$$\max_i |x_i-x_i'|=O[\max_i |x_i-x_i^0|^{k+1}],$$

where $x(x_i)$ and $x'(x_i')$ are two corresponding points, is called an analytic contact of order k (G. Fubini). Let there be given an analytic contact strictly of order k (i.e., not of order $k+1$); then, to each line-element t, tangent to V_r in x^0, there is associated, in a specified way, another line-element t^* (not necessarily tangent to V); the "linearizing transformation" $t\rightarrow t^*$ is rational of order $k+1$ and is an X_n-invariant of the given contact. One may deplore the complete absence of any bibliography; relevant references seem to be E. Čech [Publ. Fac. Sci. Univ. Masaryk no. 46 (1924); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 1 (1925), 200–204; Časopis Pěst. Mat. Fys. 74 (1950), 32–48; 75 (1950), 123–136, 137–158; MR 12, 534; 13, 158] and G. Vaona [Boll. Un. Mat. Ital. (3) 6 (1951), 293–299; MR 13, 775]. *J. L. Tits.*

Danelič, I. A. The uniqueness of certain convex surfaces in the Lobachevsky space. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 217–219. (Russian)

The author reports on results which he has obtained as extensions to Lobachevsky space of theorems which A. D. Aleksandrov and others have proved for euclidean space. The simplest type of convex surface under discussion is the "cap" (kapochka); that is a convex surface with a plane boundary curve γ whose orthogonal projection into the plane of γ is entirely contained within the domain surrounded by γ. As a typical result the first theorem may be quoted: Two isometric caps of bounded curvature in Lobachevsky space are congruent or symmetric. The notion of kapochka is generalized into "horispherical kapka" in Lobachevsky space, having n boundary curves on n horispheres, with the condition that the given surface, united with the n semi-cylinders over the curves, perpendicular to the horispheres in the direction of their axes, is a complete convex surface. Another extension is the "infinite convex surface with p points at infinity". For these the author generalizes a theorem by Busemann and Feller which states that the set of points having no normal Dupin indicatrix has measure zero.

H. Schwerdtfeger (Kingston, Ont.).

Segre, Beniamino. Dilatazioni di varietà differenziabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 249–257.

Let U and V be two real compact differentiable manifolds of class C^1, of dimensions c, d $(0\leqq c<d)$, $U\subset V$. The author shows that there exists a pair of differentiable manifolds U', V', with $U'\subset V'$, of dimensions $d-1$, d, which are determined to within a differentiable homeomorphism, such that U' is fibred over U by projective spaces of dimension $r=d-c-1$, and such that U, V are derived from U', V' by identifying all the points in each fibre. In such a case, V' is said to be derived from V by a dilatation T of the first species, with base U; the inverse $\Theta=T^{-1}$ is a continuous map of V' onto V which contracts each fibre of U' onto its base point in U. Since dilatation does not in general preserve orientability, the investigation of homology properties involves the use of coefficients mod 2. Subject to this proviso, the author shows that known results in the theory of algebraic varieties can be extended to the present situation. Thus, for instance, $\Theta(U'^k)\sim 0$ for $1\leqq k\leqq r$, while $U_{V,i}=\Theta(U'^{r+i+1})$, for $0\leqq i\leqq c$ belongs to a well-defined homology class of $r-i$ cycles (mod 2) on U. By analogy with the algebraic case, the sequence $\{U_V\}=U, U_{V,1}, \cdots, U_{V,c}$ is called the covariant sequence of immersion of U in V. As in the algebraic case, there are applications to irregular intersections; thus, if $d-c+1\leqq s\leqq d$, and $A_1, \cdots, A_s$ are $(d-1)$-cycles on V whose homology classes contain representatives passing simply through U, then the residual intersection $(A_1\cdots A_s)_V{}^U$ is homologous (mod 2) to $(A_1\cdots A_s)+\sum_{i=0}^{t} U_{V,t-i}V_i[A]$, where $V_i[A]$ is the sum of the products of $A_1, \cdots, A_s$ taken i at a time without repetitions, and $t=s-d+c$. Again, a canonical sequence can be defined for V, by taking the covariant sequence of immersion of the diagonal cycle Δ on $V\times V$ and using the natural map of Δ onto V. In a final paragraph the author mentions the possibility of dilatations of higher species, which (when they exist) can be regarded as the resultant of a dilatation T of the first species and a contraction of each S_r of U' into a subspace S_{r-k}, over which S_r is fibred by S_k. [See the following review.] *J. A. Todd.*

Segre, Beniamino. Fibrazioni differenziabili di un' r-sfera mediante k-sfere equatoriali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 383–392.

The question posed in the title is equivalent to that of fibering a real projective S_r by means of subspaces S_k. If such a fibering is possible locally, the author shows that $r\equiv k$ (mod 2^p), where p is the least integer for which $2^p>k$. A global fibering cannot exist unless $k=2^p-1$ [Steenrod and Whitehead, Proc. Nat. Acad. Sci. U.S.A. **37** (1951) 58–63; MR **12**, 847]. Much of this paper is devoted to the construction of examples; perhaps the most interesting of these establishes two projectively distinct global fiberings of the real projective S_{4m-1} by S_3.

J. A. Todd (Cambridge, England).

Muracchini, Luigi. Una classificazione delle trasformazioni puntuali di primi specie fra piani. Boll. Un. Mat. Ital. (3) **12** (1957), 204–211.

The author considers a point-to-point transformation between two projective planes π, π': $T:\pi\leftrightarrow\pi'$, of the first kind, i.e., the characteristic directions in two corresponding points O, $O'\equiv T(O')$ are distinct. First, the author considers the osculating quadratic transformations (o.q.t.) to T at O, O' and the linearizing correspondence with respect to an o.q.t., and gives some geometric properties of the o.q.t. and of the linearizing correspondences that different cases may present. These results are then exploited to give some ideas for the classification of T; as the author points out, the question as to the existence of the possible types of transformations is not considered. *V. Dalla Volta* (Rome).

See also: Functions of Real Variables: Ravetz. **Manifolds, Connections:** Blanuša.

Manifolds, Connections

Toponogov, V. A. On convexity of Riemannian spaces of positive curvature. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 674–676. (Russian)

In this note the following theorem is proved.

Let R^n be a twice continuously differentiable Riemann manifold with a complete metric of non-negative curvature. Then every triangle composed of geodesics on R_n has angles which are not smaller than the corresponding angles of the plane triangle with sides of the same length.

From the introduction.

Onicescu, O. Variétés riemanniennes multilocales. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 154–157.

Siano E_n ed E spazi euclidei ad n e ad una dimensione. Nello spazio $S_{n+1}=E_n\times E$ si conderino m punti contemporanei $M_l=P_l\times t$ $(l=1, 2, \cdots, m)$ e le m metriche

$$(1)\qquad (ds_l)^2=a_l(P_1, \cdots, P_m)(dP_l)^2+b_l(P_1, \cdots, P_m)dt^2,$$

e nello spazio $S_{nm+1}=E_n\times E_n\times\cdots\times E_n\times E$ la metrica

$$(2)\quad d\sigma_l^2=\lambda_1(dP_1)^2+\cdots+\lambda_{l-1}(dP_{l-1})^2+a_l(dP_l)^2 +\lambda_{l+1}(dP_{l+1})^2+\cdots+\lambda_m(dP_m)^2+b_l dt^2.$$

In relazione al tensore di Riemann della (2) si determinano tensori, del 2° ordine differenziali, rispetto alle forme (1). Si determina anche la relazione tra curve autoparallele dello spazio (1) e le geodetiche della varietà (2).

I risultati si generalizzano sostituendo agli spazi euclidei varietà riemanniane. *C. Longo* (Parma).

Narasimhan, M. S. The problem of limits on a Riemannian manifold. J. Indian Math. Soc. (N.S.) 20 (1956), 291–297 (1957).

Etude des prolongements self-adjoints du Laplacien dans l'espace des formes de carré sommable sur un espace de Riemann indéfiniment différentiable — L'auteur retrouve comme cas particuliers des résultats de Bochner [Duke Math. J. 3 (1937), 488–502], Gaffney [Trans. Amer. Math. Soc. 78 (1955), 426–444; MR 16, 957], Yosida [Proc. Japan Acad. 27 (1951), 540–543; MR 15, 137], Lions et Schwartz [Acta. Math. 94 (1955), 155–159; MR 17, 746]. *J. L. Lions* (Nancy).

Blanuša, Danilo. Immersion isométrique mutuelle d'espaces à courbure constante ayant une infinité de dimensions. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke 302 (1955), 87–111. (Serbo-Croatian. French summary)

The author considers several infinite-dimensional spaces, generalizations of finite-dimensional spaces, and studies the possibility of imbedding isometrically one in the other, in various combinations. The spaces considered are a Hilbert space R, a spherical space S^r of sequences ξ_i with $\sum_1^\infty \xi_i^2=1$ and metric given by the form $ds^2=r^2\sum_1^\infty d\xi_i^2$, an elliptic space E^r the same as S^r except that diametrically opposite points are identified, and a hyperbolic space H of sequences ξ_i with $\xi_1\geqq 1$ and $-\xi_1^2+\sum_2^\infty \xi_i^2=-1$ and metric given by $ds^2=r^2(-d\xi_1^2+\sum_2^\infty d\xi_i^2)$. If the symbol $X\subset_i Y$ is taken to mean that X may be imbedded isometrically in Y, then typical results of the paper can be described by such chains as

$$RC_iE^rC_iS^{r/2^{1/2}}C_iH^pC_iR,$$

when $r\leqq p$. Incidentally, the French summary referred to amounts to a translation of the entire paper with the exception of the displayed formulas. *J. W. Green.*

Moór, Arthur. Entwicklung einer Geometrie der allgemeinen metrischen Linienelementräume. Acta Sci. Math. Szeged 17 (1956), 85–120.

Consider a space of line-elements (x^i, v^i), where the x^i represent local coordinates of an underlying manifold X_n, v^i the "directional" elements of the local tangent spaces $(i, j=1, \cdots, n)$. A Finsler metric is defined on X_n when a function $F(x, v)$, satisfying certain conditions, is given, the metric tensor γ_{ij} being determined by successive derivatives of $\frac{1}{2}F^2$ with respect to v^i, v^j. The author introduces a new generalisation of such spaces by assuming that initially a metric tensor $g_{ij}(x, v)$ is given $(g_{ij}=g_{ji})$. While this gives rise to a scalar metric function $f^2=g_{ij}(x, v)v^iv^j$, the g_{ij} conversely no longer represent directional derivatives, and hence the relations $(\partial g_{ij}/\partial v^k)v^i=0$, valid in Finsler spaces, do not hold in general. {At first sight one might be inclined to regard this generalisation as being superfluous, since $\frac{1}{2}f^2$ may be differentiated to yield an alternative metric tensor in the sense of Finsler geometry. However, one is sometines confronted with problems involving the more general $g^{ij}(x, v)$, as for instance in the theory of quasi-linear partial differential equations, whose geometrisation requires techniques more general than those of Finsler geometry.} The author's approach represents a direct generalisation of the methods of E. Cartan, the covariant differential giving rise to two sets of connection parameters. It is stipulated that the covariant derivatives of the g_{ij} vanish identically. This, together with further more complicated conditions, leads to a partial determination of the connection parameters. {The author's analysis in § 2 concerning the uniqueness of the latter is erroneous; furthermore the introduction of certain inverse matrices leads to unfortunate non-explicit formulations.} Finsler spaces are characterised in terms of this geometry, while the geodesics do not in general coincide with the autoparallel curves. Curvature tensors and Bianchi identities are derived. Part II of the paper is devoted to a generalisation (by means of Lie derivatives) of the work of Knebelman [Amer. J. Math. 51 (1929), 527–564] on groups of motions in Finsler spaces. *H. Rund* (Durban).

Moór, Arthur. Über den Schurschen Satz in allgemeinen metrischen Linienelementräumen. Nederl. Akad. Wetensch. Proc. Ser. A. 60=Indag. Math. 19 (1957), 290–301.

The Geometry under consideration is a generalization of Finsler space in which the connection is a metric one in the sense that the covariant derivative of the fundamental tensor vanishes, but in which the connection parameters are not symmetric as in the theory of Cartan. [See the paper reviewed above.] The theorem of Schur for Riemannian spaces states that if the so-called Riemannian curvature calculated for a certain two-dimensional facet at a point does not depend on the facet, it is constant also from point to point.

This theorem was extended by Berwald to Finsler spaces. The author in this paper extends the theorem further to his generalization of Finsler spaces. He first introduces the notion of scalar curvature, which in this case is of two kinds. There are correspondingly two extensions of Schur's theorem. *E. T. Davies.*

Moór, Arthur. Über die autoparallele Abweichung in allgemeinen metrischen Linienelementräumen. Publ. Math. Debrecen 5 (1957), 102–118.

In a previous paper [reviewed second above] the author has developed the foundations of a theory of metric spaces, which may be regarded as representing a significant extension of Finsler geometry. Consider a manifold of line-elements (x^i, v^i), where x^i refers to a local coordinate system defined over a manifold X_n and v^i represents the "directional" elements of the local tangent spaces of X_n $(i=1, \cdots, n)$. A so-called metric tensor $g_{ij}(x, v)$ is supposed to be given: in contrast with Finsler geometry it is not assumed that the g_{ij} are derivable from a scalar function $\frac{1}{2}F^2(x, v)$ by repeated differentiation with respect to v^i, v^j. The approach of Cartan as regards the introduction of connection parameters may be generalised, i.e., a parallel displacement may be defined. In general the autoparallel curves do not coincide with the geodesics. The purpose of the present paper is the development of a theory of "autoparallel" deviation: this is the counterpart of the well-known theory of geodesic deviation, the latter having been previously developed in detail for Finsler spaces by various authors. An equation similar to the classical equation of geodesic deviation is derived for autoparallel deviation. The author also obtains a number of results concerning m-dimensional subspaces of the metric space with a view to future applications. A special study is made of the two-dimensional case for which a sufficient condition for the existence of an envelope of the set of autoparallel curves issuing from a fixed point is derived.
H. Rund (Durban).

Sulanke, Rolf. Eine Ableitung des Cartanschen Zusammenhangs eines Finslerschen Raumes. Publ. Math. Debrecen 5 (1957), 197–203.

The author considers a space of line-elements (x^i, y^i),

where the x^i represent the local coordinates of a manifold X_n, the y^i denoting "directional" elements of the tangent spaces to X_n. Such spaces of line-elements are called affinely-connected [in the sense of Varga, same Publ. 1 (1949), 7–17; MR 11, 134] when connection coefficients $C_{kh}{}^i \Gamma^*_{kh}{}^i$ are introduced. It is assumed that (a) the $\Gamma^*_{hk}{}^i$ transform like the Christoffel symbols of Riemannian geometry under point-transformations, while the $C_{kh}{}^i$ are components of a tensor; (b) the $\Gamma^*_{kh}{}^i$ and the $C_{kh}{}^i$ are homogeneous of degree 0 and -1 respectively in their directional arguments; (c) they are symmetric in k and h; (d) the $C_{kh}{}^i$ satisfy $C_{kh}{}^i(x, y)y^h=0$. Further, a metric tensor $g_{ij}(x, y)$ may be defined over the space, and it is assumed that (1) $g_{ij}=g_{ji}$, (2) det $|g_{ij}|\neq 0$, (3) $(\partial g_{ij}/\partial y^k)y^i=(\partial g_{ij}/\partial y^k)y^j=(\partial g_{ij}/\partial y^k)y^k=0$ [cf. the review third above]. The connection is said to be metric, if the corresponding covariant derivative of g_{ij} vanishes. It is shown that corresponding to a given metric tensor satisfying conditions (1), (2), (3) there exists one and only one symmetric affine connection which is also metric. This connection is, in fact, the connection of Cartan. *H. Rund* (Durban).

Watanabe, Shôji. On special Kawaguchi spaces. Tensor (N.S.) 7 (1957), 130–136.

In a special Kawaguchi space, the arc length of a curve is given by

$$\int \{A_i(x, x')x''^i+B(x, x')\}^{1/p}dt.$$

When there exists a local coordinate system in which $\partial A_i/\partial x'^j=$const. and $B=0$, the space is called locally flat. The main theorem of the paper gives necessary and sufficient conditions that a space of even dimension be locally flat. These are expressed by the vanishing of certain curvature and torsion tensors. The theorem is proved on the assumption that det $(H_{ij})\neq 0$, where $2H_{ij}=\partial A_j/\partial x'^i-\partial A_i/\partial x'^j$.

Finally, an analog to the Frenet equations of a curve is developed for a special Kawaguchi space. This theory avoids the difficulty of the case $p=3$ which occurred in a previous theory by S. Ide [Tensor (N.S.) 2 (1952), 89–98; MR 14, 586]. *C. B. Allendoerfer* (Seattle, Wash.).

See also: Algebraic Topology: Hirsch. **Relativity:** Libois.

Complex Manifolds

See: Functions of Complex Variables: Rothstein.

Algebraic Geometry

Cartier, Pierre. Calcul différentiel sur les variétés algébriques en caractéristique non nulle. C. R. Acad. Sci. Paris **245** (1957), 1109–1111.

To any algebraic variety V of dimension r over a field k, and any integer $n>0$, one can associate in a natural way a fiber space A_n, such that the fiber at any $x\in V$ is $\mathfrak{m}_x/(\mathfrak{m}_x)^{n+1}$, $\mathfrak{m}_x$ being the maximal ideal in the local ring at x; the group $G_{n,r}$ which operates on each fiber is the group of automorphisms of the algebra of polynomials in r variables, where each monomial of degree $>n$ is set equal to 0. The author first states that if E is a fiber space associated with A_n, the vector space over k of all rational everywhere defined sections of E is a birational invariant of V when V is complete; but it has since been pointed out to him that this statement is correct only if the rational representation of $G_{n,r}$ in the fibers is of "polynomial type" (i.e., if elements of $G_{n,r}$ are identified with matrices, the representation of $G_{n,r}$ should be given by matrices whose elements are polynomials in the elements of the matrices belonging to $G_{n,r}$). Let T_n be the fiber space dual to A_n, whose sections can, in a natural way, be defined as the differential operators of order $\leqq n$ on the field $k(V)$ of rational functions, which vanish on the constants. Suppose k has characteristic $p>0$, and S is a fiber subspace of T_n whose corresponding differential operators constitute an algebra $\tilde{S}$. The author's Theorem 2 states (in its correct form) that V can be given a new structure of algebraic variety (written V/S) such that the identity mapping j: $V\rightarrow V/S$ is rational, and $k(V/S)$ is the subfield of $k(V)$ on which the operators of $\tilde{S}$ vanish; S is then the kernel of the "derived" mapping $j_*:T_n(V)\rightarrow T_n(V/S)$; this result can be compared to the classical Frobenius theorem on completely integrable systems. For algebraic groups, similar ideas lead to a one-to-one correspondence between inseparable isogenies and certain subhyperalgebras [as defined previously by the author, same C. R. **244** (1957), 540–542; MR 18, 789] of the hyperalgebra of the group. Finally the author generalizes the operation $\omega\rightarrow C\omega$, which he had defined previously in the case where G is the additive group of k [ibid. **244** (1957), 426–428; MR 18, 870], to differential "forms" over V, with values in the Lie algebra $\mathfrak{g}$ of an algebraic group G. It has been pointed out to him that the result he states for the forms satisfying $C\omega=0$ is not true in general, although it holds for some types of groups G (such as the general linear group, for instance). *J. Dieudonné* (Paris).

Roquette, Peter. Über das Hassesche Klassenkörper-Zerlegungsgesetz und seine Verallgemeinerung für beliebige abelsche Funktionenkörper. J. Reine Angew. Math. 197 (1957), 49–67.

Let A be an abelian variety defined over a field k, let G be a finite subgroup of the group $A(k)$ of points on A rational over k, and let $B=A/G$ be the quotient variety. Then G is a group of translation automorphisms of the function field $K=k(A)$ and the fixed field of G is $L=k(B)$. Let $D_a(A, k)$, resp. $D_l(A, k)$, be the group of divisors on A rational over k which are algebraically, resp. linearly, equivalent to zero, and let $\hat{A}(k)$ be the group of points rational over k on the Picard variety $\hat{A}$ of A. Since G consists of translations of A, it operates trivially on $\hat{A}$ and the exact sequence $\hat{A}(k)\approx D_a(A, k)/D_l(A, k)$ yields a coboundary homomorphism δ_0: $\hat{A}(k)=H^0(G, \hat{A}(k))\rightarrow H^1(G, D_l(A, k))$. Since K/L is unramified, the kernel of δ_0 is $\varepsilon\hat{B}(k)$, where $\varepsilon:\hat{B}\rightarrow\hat{A}$ is the dual isogeny to the canonical map $A\rightarrow B=A/G$. The exact sequence $D_l(A, k)\approx K^*/k^*$ yields $\delta_1:H^1(G, D_l(A, k))\rightarrow H^2(G, k^*)$, and the kernel of δ_1 is zero by Hilbert's Theorem 90. Thus, the composed map $h=\delta_1\delta_0$ is a homomorphism of $\hat{A}(k)$ into $H^2(G, k^*)$ with kernel $\varepsilon\hat{B}(k)$. Let $C\in\hat{A}(k)$ and let $c_{\sigma,\tau}$ be a 2-cocycle representing $h(C)$. Given a constant field extension $k'\supset k$, it is clear from the naturality of h that the equation $\varepsilon D=C$ has a solution $D\in\hat{B}(k')$ if and only if the equations $c_{\sigma,\tau}=d_\sigma d_\tau/d_{\sigma\tau}$ have a solution $\{d_\sigma\}$ in k'^*. Since $\varepsilon:\hat{B}\rightarrow\hat{A}$ is surjective, such fields k' do exist, and it follows that $c_{\sigma,\tau}=c_{\tau,\sigma}$. Moreover, if we build the abelian k-algebra Γ having as k-basis a set of elements $\{u_\sigma\}$, $\sigma\in G$, with $u_\sigma u_\tau=u_{\sigma\tau}c_{\sigma,\tau}$, then the above condition on k' is equivalent with the existence of a k-homomorphism $\Gamma\rightarrow k'$. Assume now that k contains a primitive n_0th root of unity, where n_0 is the exponent of G, and let X be the kernel of

ε. Then $X \subset \hat{B}(k)$ and the function field $\bar{L}=k(\hat{B})$ is abelian over $\bar{K}=k(\hat{A})$ with Galois group X, and by Kummer theory, $X \approx \mathrm{Hom}\,(G, k)$. From the theory of algebras of type Γ one can see directly that for each $C \in A(k)$ there is a smallest field of type k', namely the "kernel field" of Γ, and this smallest k' is abelian over k with Galois group isomorphic to a subgroup of X. Thus, the way in which a point $C \in \hat{A}(k)$ decomposes in the extension $\bar{L}/\bar{K}$ is described by the cohomology class $h(C)$. This is the author's generalization of Hasse's reciprocity law. If instead of treating a rational point C of $\hat{A}$ over k we treat a generic point x of $\hat{A}$ over k, we find that the algebra Γ constructed with a representative cocycle of $h(x)$ is a field, and the extension $\Gamma/k(x)$ is isomorphic to $k(\hat{B})/k(\hat{A})$, i.e. to $\bar{L}/\bar{K}$.

When G is taken to be the group of division points of period n_0 on A, then B can be identified with A, and ε becomes simply multiplication by n_0. Thus h induces an isomorphism of the factor group $\hat{A}(k)/n_0\hat{A}(k)$ (divisor classes of K algebraically equivalent to zero modulo their n_0th powers) onto a subgroup $h(\hat{A}(k))$ of $H^2(G, k^*)$. In case k is a number field the author gives a simple proof of Weil's weak finiteness theorem by showing that there exists a finite set S of primes of k such that every element of $h(\hat{A}(k))$ can be represented by a cocycle whose elements are S-units, and consequently that $h(\hat{A}(k))$ is finite.

J. Tate (Cambridge, Mass.).

Hirai, Atuhiro. The linear equivalence theory of cycles and cycles of dimension zero on abelian varieties. J. Math. Soc. Japan 8 (1956), 180–205.

Let X^r be a cycle on an algebraic variety U^n with $r<n$. The author defines X to be linearly equivalent to 0 if $X=\sum a_iA_i\cdot(f_i)$, where the f_i are functions on U and the A_i are simple subvarieties of U. In § 1, the author discusses the operations of multiplication, intersection and algebraic projection on classes of cycles with respect to linear equivalence under certain conditions. In § 2, letting $\Gamma_1, \cdots, \Gamma_n$ be n complete nonsingular curves, he defines the module $L(\mathfrak{a})$ attached to a zero-cycle $\mathfrak{a}$ on $\Gamma_1\times\cdots\times\Gamma_n$ as follows: a function f on $\Gamma_1\times\cdots\times\Gamma_n$ is in $L(\mathfrak{a})$ if and only if the function f_i induced by f on Γ_i is such that $(f_i)>\mathfrak{a}_i$, where $\mathfrak{a}_i$ is the projection of $\mathfrak{a}$ on Γ_i. Then the author shows that the dimension $l(\mathfrak{a})$ of $L(\mathfrak{a})$ can be written as

$$(1)\qquad l(\mathfrak{a})=\sum_1^n \beta_i(\mathfrak{a})\cdot\deg(\mathfrak{a})^{n-i},\ \beta_0(\mathfrak{a})=1,$$

where the $\beta_i(\mathfrak{a})$ are integers canonically attached to $\Gamma_1\times\cdots\times\Gamma_n$ and $\mathfrak{a}$.

Let A^n be an Abelian variety, and let $\Gamma_1, \cdots, \Gamma_n$ now be n independent generic linear sections of A. Then we have a surjective map F such that $F(x_1, \cdots, x_n)=\sum_1^n x_i$ onto A. Let $\mathfrak{a}$ be a zero-cycle on A; then an expression for $l(F^{-1}(\mathfrak{a}))$ arises which is similar to (1). Finally, the author discusses the birational invariance of $l(F^{-1}(\mathfrak{a}))$ in § 4; unfortunately, his basic proposition (Prop. 1, § 4) is false. It seems to the reviewer that it is a projective invariant, rather than a birational invariant. Also his Lemma 2, § 1 seems to be incorrect.

T. Matsusaka.

See also: Differential Geometry: Segre.

NUMERICAL ANALYSIS

Numerical Methods

Kertiss, D. [Curtiss, J. H.] Monte Carlo methods for the iteration of linear operators. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 5(77), 149–174. (Russian)

A translation by V. S. Vladimirov of the article by J. H. Curtiss in J. Math. Physics 32 (1954), 209–232 [MR 15, 560].

de Greiff Bravo, Luis. Use of matrix calculus in least square corrections. Rev. Acad. Colombiana Ci. Exact. Fis. Nat. 10 (1957), 131–136. (Spanish)

An expository article.

Barlett, R. H.; Rice, M. H.; and Good, R. H., Jr. Approximations for Coulomb wave functions. Ann. Physics 2 (1957), 372–383.

This paper gives a discussion of WKB-type approximations to the solutions of the radial wave equation for a Schrödinger particle in a Coulomb field. Approximations based on Airy integrals, $(L+\frac{1}{2})$-order Bessel functions, and $(2L+1)$-order Bessel functions are considered. The results indicate that important factors in the choice of the basic function for the approximation are the poles in the effective potential. (Authors' abstract.)

J. C. P. Miller (London).

Kogbetliantz, E. G. Computation of Arctan N for $-\infty<N<+\infty$ using an electronic computer. IBM J. Res. Develop. 2 (1958), 43–53.

The author gives a survey of polynomials and rational functions which can be used to calculate arctan N on an electronic computer, taking the minimum time. The formulae are based on the Tchebyshev series for arctan (x tan 2θ), and on the continued fraction, due to Gauss, for arctan x. Special attention is given to formulae which are accurate to 8, 10, 18 or 20 significant decimal figures.

C. B. Haselgrove (Manchester).

Lyusternik, L. A. On difference approximations of the Laplace operator. Amer. Math. Soc. Transl. (2) 8 (1958), 289–351.

Translated from Uspehi Mat. Nauk (N.S.) 9 (1954), no. 2(58), 3–66 [MR 17, 303].

Orchard-Hays, Wm. Evolution of linear programming computing techniques. Management Sci. 4 (1958), 183–190.

This principally nontechnical article outlines the development, through a series of computers beginning with the Card-Programmed Calculator and ending with the 704, of machine programs for solving linear programming problems; the present codes for this are probably the most complicated in general use for a particular computational problem. The author, who was the principal architect of many of these, gives some details regarding the complications of their structure and the many forms of arithmetic he gradually found it necessary to use.

P. Wolfe (Santa Monica, Calif.).

Householder, Alston S. A survey of some closed methods for inverting matrices. J. Soc. Indust. Appl. Math. 5 (1957), 155–169.

Methods for solving equations and inverting matrices

which yield the exact solutions as a result of a finite number of arithmetic operations are called closed. Most closed methods are methods of factorization or methods of modification. If the equations to be solved are $Ax=h$, then factorization methods aim to express A as the product of two factors, each of which can be readily inverted. Methods of modification might utilize premultiplication by P with $PA=Q$ where Q is easily inverted. The author discusses various methods of factorization and modification and gives special consideration to the methods of Sherman and Morrison, Woodbury, and Kron.

P. S. Dwyer (Ann Arbor, Mich.).

Saul'ev, V. K. On the solution of the problem of eigenvalues by the method of finite differences. Amer. Math. Soc. Transl. (2) **8** (1958), 257–287.

Translated from Vyčisl. Mat. Vyčisl. Tehn. **2** (1955), 116–144 [MR **16**, 1056].

Zajta, A. Untersuchungen über die Verallgemeinerungen der Newton-Raphsonschen Wurzel-Approximation. II. Acta Tech. Acad. Sci. Hungar. **19** (1957), 25–60. (English, French and Russian summaries)

This continues part I [same Acta **15** (1956), 233–260; MR **18**, 415]. The author's aim is stated in the opening sentence „Auch diese Mitteilung verfolgt dasselbe Ziel wie die erste; Kenntnisse ueber die formalen Eigenschaften der verschiedensten Verallgemeinerungen der Newton-Raphson Formel zu einer allgemeinen und umfassenden Theorie zu entwickeln."

The article develops many threads which cannot be summarized readily. We give references to the main underlying source-papers by other authors and their reviews in MR, and give Zajta's own summary of his paper.

Both papers (part I and part II) expand results by Bodewig [Z. Angew. Math. Mech. **29** (1949); 44–51; MR **10**, 573]; by Kiss [ibid. **34** (1954), 68–69; MR **15**, 900]; and by Gornstein [Dokl. Akad. Nauk SSSR (N.S.) **78** (1951), 193–196; MR **12**, 861, note word of caution at end of Forsythe's MR review]. For definition of Konvergenzgrad=degree of convergence of a Newton approximation, see review of Zajta I in MR, above.

"This second paper is the organic continuation and completion of the previous first one. In the first chapter the verification of some basic statements is found in relation to intermediate approximate formulae in pursuance of Bodewig's definition of the degree of convergence. In the course of discussion of Gornstein's method as a principal result it is demonstrated that the application of the function $\varphi_\nu(x; a)$, set up after Gornstein's prescription, according to the formula:

$$x_{n+1}=\varphi_\nu(x_n; x_n)$$

is identical with the use of Kiss' formulae.

"In addition to the series of the Euler and of the Kiss-Gornstein formulae two further series of formulae are given in Chapter 4, which originate from the iterative transcription of the Bernoulli and Gräffe methods of approximation. Chapter 3 deals with the definition and properties of the polynomials included in the formulae, Chapter 5 with their explicit form. Further, Chapters 6 and 7 contain the investigation begun by Bodewig for the case of multiple-roots, extended on all four series of roots. The aim was to establish what transformation of the formulae is required for the preservation of the degree of convergence in the case of multiple-roots". [From the author's summary.]

A. J. Kempner (Boulder, Colo.).

Das, S. C. The numerical evaluation of a class of integrals. II. Proc. Cambridge Philos. Soc. **52** (1956), 442–448.

[For part I see Moran, same Proc. **52** (1956), 230–233; MR **17**, 901.] The author considers evaluating $I_N(h)=\int_h^\infty\cdots\int_h^\infty f(x_1, x_2, \cdots, x_N)dx_1,\ dx_2,\ \cdots,\ dx_N$, where $f(x_1, x_2, \cdots, x_N)$ is the multivariate normal density function with an arbitrary correlation matrix. For $N=2$, $I_2(h)$ is expressed as a single integral which, in turn, is numerically integrated by summing an infinite series of weighted products of pairs of truncated univariate normal distributions. Subject to specified conditions on the correlation matrix, the above approach is extended for the case $N=3$. A special method is considered when $N=1$. A discussion of the error due to the numerical integration is presented for $N=2$.

M. Muller.

Favard, J. Sur les quadratures mécaniques. Enseignement Math. **3** (1957), 263–275.

This is an outline of the treatment of the subject given by the author in his course at the École Polytechnique. The basic relation, obtained by careful integration by parts, is

$$\int_0^1 f(x)dx= \sum_{i=1}^{k} m_i f(\xi_i)+\sum_{r=2}^{n}(-1)^{r+1}\mathbf{B}_r[f^{(r-1)}(1)-f^{(r-1)}(0)]/r!+R_n,$$

where $(-1)^n n!R_n=\int_0^1 f^{(n)}(x)\overline{\mathbf{B}}_n(x)dx$. Here $\xi_1, \xi_2, \cdots, \xi_k$ are arbitrary (distinct) points in [0, 1], and the weights m_i are chosen so that the quadrature is exact for linear functions; i.e., $\sum m_i=1$; $\sum m_i\xi_i=\frac{1}{2}$. Further, $\overline{\mathbf{B}}_n(x)=\sum m_i\bar{B}_n(x-\xi_i)$; $\mathbf{B}_n=\overline{\mathbf{B}}_n(0)$, where $\bar{B}_n(x)$ coincides with the Bernoulli polynomial of degree n in [0, 1], and is defined outside this interval so as to have period 1.

Two variants of the basic relation are discussed, and various special cases are examined; e.g., the trapezoidal, Simpson and Gaussian rules.

John Todd.

Grossman, D. P. On the formulas of numerical differentiation without differences. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. **61** (1956), 30–36. (Russian)

The author proves two theorems previously announced [Dokl. Akad. Nauk SSSR (N.S.) **89** (1953), 777–779; MR **14**, 1129] and deduces some consequences from them.

P. Rabinowitz (Rehovoth).

Altman, M. On the generalisation of Newton's method. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 789–795, LXVIII. (Russian summary)

The author treats a generalization of Newton's method, $x_{n+1}=x_n-[P'(x_n)]^{-1}P(x_n)$, $n=0, 1, 2, \cdots$, for the location of a solution of $P(x)=0$. Here $P:X\rightarrow Y$ is a map between Banach spaces and P' denotes the Fréchet derivative. This paper assumes that the continuous linear operators $Q_n=P'(x_n)$ map onto Y, but permits non-zero solutions of $Q_n(x)=0$, so that the inverse Q_n^{-1} is everywhere defined, but not single-valued. (A similar discussion for the "modified" Newton's method was treated by the author in an earlier note [same Bull. **3** (1955), 189–193; MR **17**, 176].) In the case where the Q_n^{-1} are single-valued, the results reduce to theorems of L. V. Kantorovič [Trudy Mat. Inst. Steklov. **28** (1949), 104–144; MR **12**, 419] and I. P. Mysovskih [ibid. **28** (1949), 145–147; MR **12**, 419].

R. G. Bartle (Urbana, Ill.).

Altman, M. On the approximate solution of non-linear functional equations. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 457–460, XXXIX. (Russian summary)

Altman, M. Concerning approximate solutions of non-linear functional equations. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 461–465, XXXIX. (Russian summary)

Newton's method in Banach spaces ordinarily assumes that the Fréchet derivative possesses an inverse and requires the estimation of the norm of this inverse operator. These two notes consider a real-valued function F defined on a closed sphere $S(x_0, r)$ of a Banach space X and attempt to locate a solution of $F(x)=0$. Let $F'(x)$ denote the Fréchet derivative of F at x; it is a continuous linear functional. Let y be such that $F'(x_0)(y)\neq 0$. In the first note, the iteration $x_{n+1}=x_n-F(x_n)y/F'(x_0)(y)$, $n=0, 1, 2, \cdots$, is examined. The convergence of the sequence (x_n) to a solution is proved if, among other hypotheses: (i) $F'(x_0)$ has a sufficiently small norm; (ii) $F'(x)$ satisfies a Lipschitz condition at x_0; or (iii) $F''(x)$ is bounded. In the second note, a somewhat more general iteration is considered. If P is a mapping from a real Hilbert space to itself, set $F(x)=\|P(x)\|^2$. Then solutions of $P(x)=0$ can be found by solving $F(x)=0$. An example is given to show that the present method is applicable in cases where the usual Newton-Kantorovič method fails. The methods in the present papers are special cases of the case treated in the paper reviewed above. *R. G. Bartle.*

Altman, M. On the approximate solutions of operator equations in Hilbert space. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 605–609, LII. (Russian summary)

Altman, M. Concerning the approximate solutions of operator equations in Hilbert space. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 711–715, LXII–LXIII. (Russian summary)

Altman, M. A note on the approximate solutions of non-linear operator equations in Hilbert space. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 783–787, LXVII–LXVIII. (Russian summary)

Altman, M. Connection between the method of steepest descent and Newton's method. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 1031–1036, LXXXVI. (Russian summary)

These four notes apply the methods of the papers reviewed above to the approximate solution of the equation $P(x)=0$, where P is a mapping of a closed sphere $S(x_0, r)$ of a real Hilbert space H, into H. It is assumed that the Fréchet derivative P' exists and has a bounded inverse on this sphere. In the first paper, sufficient conditions are given for the convergence of the iteration

$$\text{(A)}\qquad x_{n+1}=x_n-Q(x_n)\frac{\|P(x_n)\|^2}{2\|Q(x_n)\|^2}\ (n=0, 1, 2, \cdots),$$

where $Q(x)=\bar{P}'(x)P(x)$, the bar denoting the adjoint operator. It is also assumed either that $Q'(x)$or $P''(x)$ is bounded on the sphere. The second and third notes deal with the iteration

$$\text{(B)}\qquad x_{n+1}=x_n-P(x_n)\frac{\|P(x_n)\|^2}{2(P'(x_n)P(x_n), P(x_n))}\qquad (n=0, 1, 2, \cdots),$$

under similar hypotheses. Under certain conditions, it is seen that a more rapid convergence occurs in (B) than in (A). The fourth note shows that the method of steepest descent [cf. L. V. Kantorovič, Uspehi Mat. Nauk (N.S.) 3 (1948), no. 6(28), 89–185; MR 10, 380], as applied to the linear equation $Ax-y=0$, with A symmetric, is a special case of method (B) with $P(x)=Ax-y$. Returning to non-linear equations, a process obtained from (B) by dropping the factor of 2, which is obtained from the notes in the preceding review by taking $F(x)=\|P(x)\|$, is examined under appropriate hypotheses on the existence or boundedness of $P'(x)$, $P'(x)^{-1}$, $P''(x)$ and $Q'(x)$ on the sphere. *R. G. Bartle* (Urbana, Ill.).

Altman, M. An approximate method for solving linear equations in a Hilbert space. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 601–604, LI–LII. (Russian summary)

The author discusses the solution of the linear equation $Ax=y$, where A is an additive homogeneous transformation in Hilbert space with domain $D(A)$, such that $\|Ax\|\geq c\|x\|$ for all x in $D(A)$. Results of a previous paper by the author [same Bull. 5 (1957), 365–370; MR 19, 581] are applied to give an iterative method for the approximate solution, based on approximate orthogonal projection on finite dimensional spaces. The method is connected with the Gauss-Seidel process. A recurrent formula for the error estimate is also given. *R. G. Bartle.*

Fisher, Michael E. On the continuous solution of integral equations by an electronic analogue. I. Proc. Cambridge Philos. Soc. 53 (1957), 162–174.

The integral equation $y(x)=f(x)+\lambda\int_0^1 K(x,t)y(t)dt$ is treated by "quantization" and iteration. This means that the integral operator is replaced by a $N\times N$ matrix, while $y(x)$ and $f(x)$ are substituted by column-vectors with N components each. The iteration is the one by Gauss-Seidel with preference for the method of successive displacements. The number N is between 10 and 100; $N=32$ is preferred.

An electronic analogue computer with electrostatic storage as well as electronic samplers, adders and multipliers is used to perform the elementary operations involved. The procedure is illustrated by an example pertaining to the kernel $K=x-t$. An error analysis discusses various sources of error, especially the influence of noise. *H. Bückner* (Schenectady, N.Y.).

Vilner, I. A. La nomographie stéréoscopique et l'anamorphose d'espace avec une échelle donnée. Ukrain. Mat. Ž. 9 (1957), 121–133. (Russian. French summary)

Dans ce mémoire nous introduisons les nomogrammes aux points alignés stéréoscopiques de l'espace de trois dimensions et nous étudions les conditions suffisantes, auxquelles doit satisfaire une équation pour admettre une nomogramme stéréoscopique. *Du résumé de l'auteur.*

Fröberg, Carl-Erik; and Wilhelmsson, Hans. Table of the function $F(a, b)=\int_0^a J_1(x)(x^2+b^2)^{-\frac{1}{2}}dx$. Kungl. Fysiogr. Sällsk. i Lund Förh. 27 (1957), 201–215.

This paper gives 6-decimal values of the function

$$F(a, b)=\int_0^a J_1(x)(x^2+b^2)^{-\frac{1}{2}}dx=\int_0^{a/b} J_1(bx)(x^2+1)^{-\frac{1}{2}}dx$$

for a, b each 0(0.1)2(0.2)10. It is stated that 5-point

Lagrangean Interpolation is adequate everywhere except for $0 \leqq a$, $b \leqq 1$. The auxiliary function $f(a, b)$, where $F(a, b) = \frac{1}{2}[(a^2+b^2)^{\frac{1}{2}} - b] - [f(a, b)]^3$ is therefore given, also to 6 decimals, for a, b each 0(0.1)1. *J. C. P. Miller.*

Longman, I. M. Tables for the rapid and accurate numerical evaluation of certain infinite integrals involving Bessel functions. Math. Tables Aids Comput. 11 (1957), 166–180.

This interesting paper is concerned with evaluation of the integrals $\int_0^\infty J_0(x)g(x)dx$ and $\int_0^\infty J_1(x)h(x)dx$. To use the first as an example, write $x_0=0$, and x_i for the ith zero of $J_0(x)$. Then the integral may be written

$$\sum_1^\infty \int_{x_{i-1}}^{x_i} J_0(x)g(x)dx.$$

If $g(x)$ is of one sign, the terms in the series alternate in sign and may often be summed by application of Euler's transformation.

The paper gives tables for use in evaluating the first 20 terms of the series, using 16-point Gaussian quadrative in each interval. Values of x and $J_0(x)$ corresponding to the Gauss points are to 10 decimals. The values of x_i are also given. A precisely similar table is given for the second integral involving $J_1(x)$.

A numerical example illustrates the method.

J. C. P. Miller (London).

See also: Functions of Complex Variables: Pergamenceva. **Special Functions:** Kreyszig. **Partial Differential Equations:** Laasonen. **Statistics:** Tukey.

Computing Machines

Yanov, Yu. I. On equivalency and transformations of program schemes. Dokl. Akad. Nauk SSSR (N.S.) 113 (1957), 39–42. (Russian)

In programming for universal automatic computing machines use is made of (logical) program schemes (p.s.). Since a p.s. is non-uniquely determined by an algorithm, questions of equivalence and also of identical transformations arise. In the present article p.s. are considered as defined by a sequence of operators and logical conditions depending on the values of logical (two-valued) variables (taking the values 0 and 1, where 0 corresponds to falsity and 1 to truth). Here the operators are regarded as elementary objects to which is assigned the definite property of changing the values of the logical variables. (From the introduction). *S. Kulik* (Columbia).

★**Booth, Kathleen H. V. Programming for an automatic digital calculator.** Academic Press Inc., New York; Butterworths Scientific Publications, London, 1958. vii+238 pp. $7.50.

This book gives, paradoxically, a "complete" introduction to programming for a digital computer — from a limited point of view. There are two schools of thought as to how digital computer programming should be taught: 1) as an extension of the detailed techniques used in operation of desk calculators, with emphasis on step-by-step "coding", listing of instructions one-by-one in sequence; 2) as an extension of the algebraic and notational languages of mathematics, with emphasis on the ideas of automatic translation from these languages into the detail instruction-by-instruction languages of the computer. The reviewer is biased towards the latter approach. This book is a competent exponent of the first school of thought.

The author lists, and in some cases develops, computer codes—sequences of instructions — for many standard digital computer processes. The language is the basic machine language of the APEXC computer at Birkbeck College, London University. This machine is an 8,192-word magnetic drum storage computer, each word consisting of 32 binary digits. The usual complement of "one-plus-one address" operations — minus multiplication and division, which are subroutines — is built in. Input-output is by paper tape. The U.S. machines most nearly corresponding in structure are the Royal-McBee LGP-30 and the Bendix G-15.

Detailed subroutines are given for input-output conversion between decimal and binary number systems, division, multiplication, square root, number normalization, finding the maximum, etc. Detailed programs for matrix multiplication, simultaneous equation solution, iteration for largest eigenvalue, etc. are also given. A program for simple machine language translation — French to English — is displayed, but no results described.

A detailed account of an interpretive program and of the checking techniques (changed word post mortem and tracing routine) used on the APEXC is given.

The notation is somewhat different than usual and the failure to use flow diagrams makes study difficult. No discussion of automatic programming (translators and compilers) is given. Nevertheless, this book takes its place with a sequence of British programming texts that give a detailed account of the experience of a group with one particular machine. For a professional programmer or machine user who will be involved with construction of programs to be used many times, this book, despite notational difficulty, should be skimmed, and parts read in detail. For individuals more interested in "modern" techniques of getting problems solved using digital computers, it will not be as useful. *J. W. Carr*, III (Ann Arbor, Mich.).

Hatton, D. E.; and Ward, J. R. An electronic analogue computer. Commonwealth of Australia. Dept. of Supply. Aero. Res. Comm. Rep. ACA-57 (1955), 34 pp.

The paper outlines the theory of electronic analogue computation and of the application to ordinary differential equations. Special emphasis is on the description of an analogue computer of the Department of Aeronautics, University of Sydney. The instrument uses D. C. amplifiers. They are described in detail together with operational circuits and auxiliary equipment. Test results pertaining to linear and nonlinear differential equations of controlsystems are presented. Appendices refer to an analysis of the computer operations, to computational accuracy and to measuring methods.

H. Bückner (Schenectady, N.Y.).

Cahn, C. R. Solution of algebraic equations on an analog computer. Rev. Sci. Instrum. 27 (1956), 856–858.

The paper presents a method for finding a real or complex root of an algebraic equation on an electronic differential analyser; it consists of setting up the corresponding homogeneous linear differential equation with constant coefficients. *W. Freiberger* (Providence, R.I.).

See also: Theory of Algebraic Numbers: Cohn. **Special Functions:** Newman. **Numerical Methods:** Kogbetliantz; Orchard-Hays; Fisher. **Elasticity, Plasticity:** Targoff. **Fluid Mechanics, Acoustics:** Koga. **Quantum Mechanics:** Worsley. **Information and Communication Theory:** Panov.

PROBABILITY

Handscomb, D. C. On the random disorientation of two cubes. Canad. J. Math. 10 (1958), 85–88.

The relative orientation of two identical symmetrical bodies is determined by the rotation about some axis through the center of gravity of one body such that it is brought into the same orientation as the other body. The author considers two cubes with independent random orientations and determines the distribution of the smallest angle of rotation needed to bring the bodies into the same orientation. *E. Lukacs* (Washington, D.C.).

Mackenzie, J. K. Second paper on statistics associated with the random disorientation of cubes. Biometrika 45 (1958), 229-240.

Mackenzie and Thomson [Biometrika 44 (1957), 205–210; MR 19, 190] estimated the distribution discussed in the preceding review by a Monte Carlo method. In the present paper the author derives the exact distribution of the smallest angle of rotation as well as the distribution of the least acute angle between the edges of the two cubes. Tables of these distributions are also given. *E. Lukacs* (Washington, D.C.).

Trybula, S. On parameters of the distribution of a random variable with cyclically ordered values. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 863–866, LXXIV. (Russian summary)

Statistical parameters (average, spread) for distributions on the unit circle. *H. P. McKean, Jr.*

Tukey, John W. On the comparative anatomy of transformations. Ann. Math. Statist. 28 (1957), 602–632.

The author considers the family of transformations $z=(y+c)^k$, their limits, including $z=\log(y+c)$ and $z=e^{my}$, and their uses in obtaining approximate distributions. He considers various properties of these transformations as a system of transformations, including practical means of representing the system graphically. For analytic reasons he requires that $c\geqq 0$, and he claims that most applications will have $p\leqq 1$, but that restriction is inessential. The uses of these transformations to obtain approximate normality of the transformed variable are discussed, and an example is given. He defines the strength of a transformation to be $\Delta \log(dz/dy)/\Delta \log y$ and points out that transformations of similar strength will have similar properties. The bulk of the paper is concerned with the topological and geometric properties of the transformations considered. *H. Rubin* (Eugene, Oreg.).

Rios, Sixto. On regression lines. Trabajos Estadist. 8 (1957), 147–156. (Spanish. English summary)

The author proposes various definitions of a regression line. He considers random variables ξ, η whose joint distribution is given by a probability density $f(x, y)$. He writes $H[g(x), \varepsilon]$ for prob$[|y-g(x)|<\varepsilon]$, and defines the functional $G[g(x)]$ to be the limit, if it exists, of $H[g(x), \varepsilon]/(2\varepsilon)$ as $\varepsilon\to 0$. "The function $g(x)$ which makes this functional a maximum gives us the regression line of η upon ξ." Tacitly assuming $f(x, y)$ to be continuous, the author observes that the regression line L maximizes the curvilinear integral $\int_L f(x, y)dx$, and that the ordinate of L at x is the mode (if unique) of the distribution of y for this (fixed) x. He also considers an nth degree parabola of regression, which is defined as above, except that $g(x)$ is now restricted to be a polynomial of the nth degree, and he extends his discussion to cover a generalized regression line $y=C(x)$, where $C(x)$ is a "central value" of y obtained by minimizing a generalized "dispersion". *H. P. Mulholland* (Exeter).

Solari, M. E.; and Anis, A. A. The mean and variance of the maximum of the adjusted partial sums of a finite number of independent normal variates. Ann. Math. Statist. 28 (1957), 706–716.

Let $[X_i]$ be a sequence of mutually independent random variables with a common distribution having mean 0 and variance 1. Let $\bar{X}_n=1/n \sum_{i=1}^n X_i$, and $S_r'=\sum_{i=1}^r X_i-r\bar{X}_n$ $(r=1, 2, \cdots, n)$. Let $U_n=\max[S_1', S_2', \cdots, S_n']$, and $V_n=\min[S_1', S_2', \cdots, S_n']$. Feller [Ann. Math. Statist. 22 (1951), 427–432; MR 13, 140] found the asymptotic distribution of the range U_n-V_n. In the present paper the common distribution is assumed normal, and the distribution of the maximum U_n is found. Formulas for the first two moments are given. Comparison of the exact values with the asymptotic values for values of n up to 150 are given. The approximation of the first two moments is poor for these values, but because of cancellation of errors, the approximation for the variance is very good even for n as small as 10. *J. L. Snell.*

Halphen, Étienne. L'analyse intrinsèque des distributions de probabilité. Publ. Inst. Statist. Univ. Paris 6 (1957), 79–159.

The theme of this paper is the properties of a discrete probability space that are invariant under permutation of its points. The general moment of the probability itself $\Psi(t)$ is stressed as one such invariant that determines all others. The obvious fact that the Ψ of a cartesian product is the product of the corresponding Ψ's is brought out and exploited. The derivatives $\Psi^{(r)}(0)=\sum p_i(\log p_i)^r$ are studied. In particular, many now familiar properties of $\Psi'(0)$, now called the Shannon-Wiener information, are derived. There is a historically remarkable anticipation here, for this posthumous paper was written in 1939–40. *L. J. Savage* (Chicago, Ill.).

Dwass, Meyer; and Teicher, Henry. On infinitely divisible random vectors. Ann. Math. Statist. 28 (1957), 461–470.

Let X be an infinitely divisible random vector. The paper gives a necessary and sufficient condition for the existence of a constant matrix A and a random vector Y with independent infinitely divisible components such that X and AY have the same distribution. Connections with the multivariate Poisson distribution and certain stochastic processes are pointed out. *G. E. Noether.*

Lukacs, Eugene. Correction to "On certain periodic characteristic functions". Compositio Math. 13 (1958), 127.

These are the corrections contained in the letter to the reviewer [MR 18, 769] of the article in same Compositio 13 (1956) 76–80.

Steck, George P. Limit theorems for conditional distributions. Univ. California Publ. Statist. 2 (1957), 237-284.

It is shown that, under certain conditions of equicontinuity, the convergence in law of a sequence of random vectors (U_s, V_s), $s=1, 2, \cdots$, to a random vector (U, V) implies the (weak) convergence of the conditional

distribution function of U_s, given $V_s=v$, to the conditional distribution function of U, given $V=v$, uniformly for v in a bounded set. The general results are applied to obtain asymptotic distributions of statistics related to Chi square and to construct approximately similar tests. In particular, the statistic $\chi^2=\sum_{j=1}^n (X_{s,j}-sp_{s,j})^2/(sp_{s,j})$, where $(X_{s,1}, \cdots, X_{s,n})$ has the multinomial distribution with probabilities $p_{s,1}, \cdots, p_{s,n}$, and $\sum_{j=1}^n X_{s,j}=s$, is shown to be asymptotically normally distributed if $0<\alpha<np_{s,j}<\beta<\infty$ for all n, s, j and $s^2/n\to\infty$ as $s\to\infty$ (special case of Theorem 3.1), as well as under certain alternative conditions (Theorem 3.3). [For still different sufficient conditions for the asymptotic normality of χ^2, see S. H. Tumanyan, Dokl. Akad. Nauk SSSR (N.S.) **94** (1954), 1011–1012; Teor. Veroyatnost. i Primenen. **1** (1956), 131–145; MR **15**, 806; **19**, 467.] The author has informed the reviewer that his statement (p. 239) to the effect that Theorem 3.3 implies one of Tumanyan's theorems is due to an oversight, but that a change in the proof of the former theorem will yield the latter. *W. Hoeffding.*

Breiman, Leo. A counterexample to a theorem of Kolmogorov. Ann. Math. Statist. **28** (1957), 811–814.

A. Kolmogorov [Math. Ann. **99** (1928), 309–319; **102** (1929), 484–488] proved that if $X_1, X_2, \cdots$ are mutually independent random variables with zero means, then $n^{-1}\sum_{k=1}^n X_k\to 0$ in probability as $n\to\infty$ if and only if: (i) $\sum_{k=1}^n P(|X_k|\geq n)\to 0$; (ii) $n^{-1}\sum_{k=1}^n EX_{nk}\to 0$; and (iii) $n^{-2}\sum_{k=1}^n E(X_{nk}-EX_{nk})^2\to 0$; where $X_{nk}=X_k$ if $|X_k|<n$, and $=0$ if $|X_k|\geq n$. In the same paper and in later books by various authors, it is stated that the theorem remains true if (iii) is replaced by (iii′) $n^{-2}\sum_{k=1}^n EX_{nk}^2\to 0$. The author gives an example in which conditions (i), (ii), (iii) are satisfied but not (iii′). *W. Hoeffding.*

Yaglom, A. M. Correlation theory of processes with random stationary nth increments. Amer. Math. Soc. Transl. (2) **8** (1958), 87–141.

Translated from Mat. Sb. N.S. **37(79)** (1955), 141–196; MR **17**, 167.

★ **Pollaczek, Félix. Problèmes stochastiques posés par le phénomène de formation d'une queue d'attente à un guichet et par des phénomènes apparentés.** Mémor. Sci. Math., no. 136. Gauthier-Villars, Paris, 1957. 123 pp.

This is a thorough study of problems associated with a single server stochastic system. These problems are mainly those of delay when a waiting line is permitted, service being in order of arrival without defections from the waiting line. The case, arising in air traffic, where a delayed arrival in joining service suffers an additional (random) delay due to this very fact, as well as the usual case where joining service is instantaneous, is considered and is modified later to permit only quantized delays, integral multiples of some appropriate time unit. But also the problem of the distribution of the "busy period", and the problem of loss when arrivals finding the server busy are dismissed (without effect on future arrivals), appear in their proper places.

As usual, the service time distribution, which is of general character throughout, is supposed independent of the entities served. The arrivals themselves are first supposed individually and collectively at random, that is, with a Poisson distribution in any time interval, but this is relaxed to the single requirement that they form a renewal process (a "general input" in D. G. Kendall's terminology) specified by the common interarrival time distribution, again of general character. There is also a chapter devoted to the "Bernoulli distribution" of arrival times (a subject the author has made almost his own), where a fixed number of arrivals appear at random in a given time interval, a condition again natural in the study of air traffic.

The treatment throughout employs complex variable theory and to an expert like the author may be regarded, as he says, as a set of exercises in the elementary part of that theory, (which should test the skill of most novices). This of course conditions the generality of the various distribution functions in question but, as the author says, in a way not serious for any of the known applications. A brief résumé of each of the nine chapters follows.

The first chapter formulates for Poisson arrivals the delay distribution function of a given (numbered) arrival by finding what is effectively a recurrence in Laplace transforms, which is readily adapted to arbitrary initial conditions (the delay of the first arrival). The assumption of stationarity (statistical equilibrium) then leads at once to (the Laplace transform of) the limiting delay distribution, of central technical interest. Technicians may stop here, but the author goes on to prove stationarity through the generating function of delays of all arrivals. The remainder of the chapter is devoted to finding the delay of an arrival at an arbitrary epoch.

The second chapter exemplifies the development of the first by choosing specific service time distributions, first as a single exponential function, next as a sum of exponentials, next as a single step function, and finally as a sum of step functions. An interesting result is that the single step function achieves the minimum mean delay.

The third chapter formulates delays for a Bernoulli distribution of arrivals, noting first the details of formulation necessary to show the Poisson distribution of arrivals as a limiting case, and then giving the development corresponding to that of the first chapter. Finally an asymptotic formula for the difference of the stationary delay distributions for the two cases (Bernoulli and Poisson arrivals) is obtained.

The fourth chapter takes up the effects of the additional delay incurred in joining service, and the fifth the modifications to quantize all delays. The sixth chapter considers the formulation of joint delay distributions of several variables, such as those of two given arrivals, say the nth and the mth, $m=n+n_1$, reaching results such as the size of n_1 to approach independence. The seventh chapter gives the generalization to arrivals constituting a renewal process, which in the author's formulation proceeds easily. The resulting formulas are exemplified by taking either the arrival or the service distribution function as a sum of gamma type distributions; an interesting aspect of the former is that when the sum is of Erlangian form, that is, when the arrival distribution corresponds to the sum of s random variables, each with the same exponential distribution, the delay distribution is the same as that for s servers with Poisson arrivals assigned to the servers in sequence on arrival, a problem examined by the author in his earliest paper on the subject [Math. Z. **32** (1930), 64–100, 729–750, 796]. The last two chapters consider the busy period and the no-delay case, and a note deals with the solution of an integral equation needed in the seventh chapter.

The uniform and careful treatment of a live subject makes the booklet an attractive addition to the dis-

tinguished series of which it is a part, though some mathematicians might prefer to see a freer use of the theory of the Laplace transform. A minor cavil is that the author's use of abbreviations for important terms (f.r. for fonction de répartition, etc.) and his reluctance to write equations over again, preferring references back, disturb the reader's ease as well as the expository flow.

J. Riordan (New York, N.Y.).

★ **Feller, William. On boundaries defined by stochastic matrices. Applied probability.** Proceedings of Symposia in Applied Mathematics, Vol. VII, pp. 35–40. McGraw-Hill Book Co., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1957. $5.00.

An operator A defined on a space E may define a boundary for E, and, if A is unsymmetric, the adjoint of A may define a different boundary of E. The situation is illustrated for the case of an enumerable space E and for A given by a quasi-stochastic matrix $A(i,j)$: $A(i,j)\geqq 0$ and $\sum_j A(i,j)\leqq 1$. Let P denote the totality of all non-negative vectors $x=\{x(i)\}$ satisfying $x=Ax$ and $\|x\|=\sup_i|x(i)|\leqq 1$. The extremal points, in the sense of Krein and Mil'man, of the positive cone P coincide with the sojourn probabilities $s_\Gamma=\{s_\Gamma(i)\}$: $s_\Gamma(i)$ is the probability that, starting from i, the system (governed by the Markov chain $A(i,j)$) will, after a finite number of steps, pass into Γ and stay there forever. Let all the states i be transient ($\sum_n A^n(i,i)<\infty$ for $i=1,2,\cdots$) and let Γ be a sojourn set in the sense that $s_\Gamma\not\equiv 0$. Then there exists a monotonically decreasing sequence of sets $\{\Gamma_n\}$ contracting to the empty set such that for each i we have $s_\Gamma(i)=s_{\Gamma_n}(i)$. Thus the net $\{\Gamma_n\}$ will define an escape route to infinity, and hence each Γ_n is a vicinity of the exit boundary point of E corresponding to this escape route.

Next let $a=\{a(i)\}$ be a solution of $a=aA$ such that $a(i)\geqq 0$, $\sum_i a(i)=1$. Then the strictly stochastic matrix $A^*(i,j)=A(i,j)a(j)/a(i)$ gives a (Kolmogorov) time reversion of A. As above, $A^*(i,j)$ will define another boundary of E, and it is proved that this boundary is defined independently of the choice of the vector a. In this way we may endow E with boundaries. The boundary defined by A^* gives the entrance boundary corresponding to A. Contrary to the classical diffusion cases, there is absolutely no connection between the two boundaries. Thus, for example, three points of the exit boundary together may form two points of the entrance boundary.

It is indicated that the notion of the two boundaries is very useful for the discussion of Kolmogoroff differential equations, if we pass from the time discrete case $A^n(i,j)$ to the continuous time parameter case $A(i,j;t)$, $t\geqq 0$.

K. Yosida (Tokyo).

Mott, J. L.; und Schneider, Hans. Matrix norms applied to weakly ergodic Markov chains. Arch. Math. **8** (1957), 331–333.

Let ν be a multiplicative norm on the subalgebra of the algebra of real matrices of some fixed order over the reals. Let $n\to\infty$. An element of this subalgebra is stable if it has identical rows, and a sequence $P_1, P_2, \cdots$ of its elements is weakly ergodic if there exist stable Q_n such that $P_1P_2\cdots P_n-Q_n\to 0$. The authors prove that if there exists a sequence $T_1, T_2, \cdots$ of stable matrices and a norm ν such that $\prod_{i=1}^{\infty}\nu(P_i-T_iP_i)\to 0$, then the sequence $P_1, P_2, \cdots$ is weakly ergodic. The proof follows from a special case of a Bourbaki's theorem; namely, if $\nu(P_n)\to 0$ then $P_n\to 0$. Examples are given. *M. Loève.*

Mott, J. L. Conditions for the ergodicity of non-homogeneous finite Markov chains. Proc. Roy. Soc. Edinburgh. Sect. A. **64** (1957), 369–380.

"In this note, we study the asymptotic behavior of a product of matrices $P^{(n)}=\prod_{j=1}^{n}P_j$, where P_j is a matrix of transition probabilities in a non-homogeneous finite Markov chain. We give conditions that (i) the rows of $P^{(n)}$ tend to identity and that (ii) $P^{(n)}$ tends to a limit matrix with identical rows". [Author's summary.]

J. Wolfowitz (Ithaca, N.Y.).

Kennedy, Maurice. A convergence theorem for a certain class of Markoff processes. Pacific J. Math. **7** (1957), 1107–1124.

The following absorption problem is considered. Let $t_i i\geqq 1$ be points in a compact metric space (Ω,ρ). Let $A_i i\geqq 1$ be continuous transformations of Ω into itself, such that A_i maps every open sphere with centre t_i into itself, and $\lim_{n\to\infty}A_i^n t=t_i$ for all $t\in\Omega$. For given probabilities, $0\leqq p_i(t)\leqq 1$ which are continuous functions on Ω satisfying $\sum_{i=1}^{\infty}p_i(t)=1$, a Markov process $\{X_n:n\geqq 1\}$ is defined by $\Pr[X_n=A_it|X_{n-1}=t]=p_i(t)$ and a given initial distribution μ. Thus there is a transformation T on the space of signed (Borel) measures on Ω such that $T^n\mu$ represents the probability measure of X_n. Let t_i, $i=1,2,\cdots,m<\infty$ be absorption points; i.e., $p_i(t_i)=1$. Under assumptions which insure that, for any intial state, some absorption point may be reached, the author proves the existence of the limiting (in a weak-star sense) distribution $T_\infty=\lim_{n\to\infty}T^n\mu$. A representation of T_∞ is also given. The above result generalizes that of Bellman et al. [Rand Corporation Res. Memo. no. RM **878** (1952)], and Karlin [Pacific J. Math. **3** (1953), 725–756; MR **15**, 450]; the approach of the latter author is employed. *R. Pyke.*

Jacobs, Konrad. Markoffsche Prozesse mit monomialer Selbststeuerung. Arch. Math. **8** (1957), 298–308.

Consider the discrete Markov process determined by the family of d-dimensional transition matrices $\{P(n,m)$: $n\geqq 0, m\geqq 0\}$, which satisfies $P(n,m)=P(n,j)P(j,m)$ for all $m\leqq j\leqq n$. The author studies the asymptotic behavior of $P(n,0)$, as $n\to\infty$, when there exists a transformation f of the appropriate space of matrices into itself, satisfying $P(n+1,n)=fP(n,0)$ for all n (f is called 'Selbststeuerung'). It is proven that when f has the particular form $fP=A_0PA_1P\cdots A_{k-1}PA_k$ for fixed matrices A_i, the sequence $P(n,0)$ is asymptotically periodic with period $\leqq(d!)^2$.

R. Pyke (Stanford, Calif.).

Castoldi, Luigi. Successioni stocastiche di tipo Markoviano generalizzato: le "cascate" finite di Markov. Atti Accad. Ligure **13** (1957), 45–54.

The author defines a generalized Markov chain to be a Markov chain in which the state space changes in a definite manner. He gives as an example a genetic model for the sex-ratio. He makes the claim, which appears to be biologically untenable, that this model explains the human sex-ratio. Erratum: The characteristic equation (24) should be $\lambda^3-\lambda^2=0$. *H. Rubin* (Eugene, Oreg.).

Sevastjanov, B. A. An ergodic theorem for Markov processes and its application to telephone systems with refusals. Teor. Veroyatnost. i Primenen. **2** (1957), 106–116. (Russian. English summary)

A rather complicated set of conditions is given insuring the existence of a unique ergodic stationary distribution for a Markov process homogeneous in time. The proof is

an extension of Markov's method. "This theorem is used to substantiate Erlang's formula for the case of an arbitrary distribution law for the duration of conversation with a finite mean value". *K. L. Chung* (Syracuse, N.Y.).

Kanngiesser, W. Grenzwertsätze für verschwindende Übergangswahrscheinlichkeiten. Mitteilungsbl. Math. Statist. 8 (1956), 15–31, 141–153, 177–191.

This concerns Markov chains with stationary transition and two states, 0 and 1. The Bernoulli problem of the probability of x ones in the first n trials and the Pascal problem of the probability of the occurrence of the mth one at the $(m+z)$th trial are studied by direct computations and some numerical tables are given. The comprehensive work by Dobrusin [Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 291–330; MR **15**, 329] is on a different level and not mentioned. *K. L. Chung* (Syracuse, N.Y.).

Harris, T. E. Transient Markov chains with stationary measures. Proc. Amer. Math. Soc. 8 (1957), 937–942.

Let (p_{ij}) be the stationary transition matrix of an irreducible, transient, discrete parameter Markov chain and consider the system of equations $u_i=\sum_j u_j p_{ji}$ for all i. A necessary condition that this has a positive solution is that there is an infinite sequence of distinct states $(\cdots, i_{n+1}, i_n, \cdots)$ such that $p_{i_{n+1}i_n}>0$. A sufficient condition is that for each i, $p_{ji}=0$ except for a finite set of j. A more inclusive and involved sufficient condition is proved. *K. L. Chung* (Syracuse, N.Y.).

Ambarcumyan, G. A. Moments of distribution of a Markov process. Akad. Nauk Armyan. SSR. Izv. Fiz. Mat. Estest. Tehn. Nauki 9 (1956), no. 5, 25–41. (Russian. Armenian summary)

Karlin, S.; and McGregor, J. L. The differential equations of birth-and-death processes, and the Stieltjes moment problem. Trans. Amer. Math. Soc. 85 (1957), 489–546.

Proofs of some of the results announced earlier [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 387–391; MR **17**, 166]; an outline was given in section (2) of the review cited above. Let $\lambda_n>0$ for $n\geqq0$, $\mu_n>0$ for $n\geqq1$, $\mu_0\geqq0$; define the matrix $A=(a_{ij})$ by $a_{nn}=-(\lambda_n+\mu_n)$, $a_{n,n+1}=\lambda_n$, $a_{n,n-1}=\mu_n$, $a_{ij}=0$ for $|i-j|>1$; and consider matrices $P(t)$ $(t\geqq0)$ satisfying some or all of the conditions (1) $P'(t)=AP(t)$, (2) $P'(t)=P(t)A$, (3) $P(0)=I$, (4) $P(t)\geqq0$, (5) $\sum_j P_{ij}(t)\leqq1$, (6) $P(s+t)=P(s)P(t)$. Results will be stated only for $\mu_0=0$ (they are less complete for $\mu_0>0$), not necessarily in their most general form. (A): Define polynomials Q_n inductively by $Q_{-1}(x)=0$, $Q_0(x)=1$, $\lambda_nQ_{n+1}(x)-(\lambda_n+\mu_n+x)Q_n(x)+\mu_nQ_{n-1}(x)=0$ $(n\geqq0)$, and put $\pi_0=1$, $\pi_n=\lambda_0\lambda_1\cdots\lambda_{n-1}/\mu_1\mu_2\cdots\mu_n$ $(n\geqq1)$. Let M be the class of regular measures ψ on $(-\infty,\infty)$ such that $\pi_i\int Q_i(x)Q_j(x)d\psi(x)=\delta_{ij}$. Then $\psi\in M$ if and only if $\int x^n d\psi(x)=c_n$, where the moment constants c_n depend only on the λ_i and μ_i and can be computed recursively from the equations $\int Q_0d\psi=1$, $\int Q_nd\psi=0$ $(n\geqq1)$. The Stieltjes moment problem for the c_n is solvable, so that M contains at least one positive ψ (necessarily of total mass 1) supported by the half-line $x\geqq0$. Every solvable Stieltjes moment problem with $c_0=1$ can be derived in this fashion from some "birth-and-death" matrix A with $\mu_0=0$. (B): Let $\psi\in M$ be supported by some half-line $x\geqq x_0>-\infty$, and define $P(t)$ by (7), $P_{ij}(t)=\pi_j\int e^{-xt}Q_i(x)Q_j(x)d\psi(x)$. Then $P(t)$ satisfies (1)–(3); if ψ is positive, then (4) holds in the strong sense that $P_{ij}(t)>0$ for $t>0$, (5) holds if ψ is supported by $x\geqq0$, and (6) holds if and only if ψ is an extremal solution of the moment problem (i.e. $\pi_n^{\frac12}Q_n(x)$ is a complete orthonormal set in $L_2(d\psi)$). Thus any extremal solution ψ of the Stieltjes moment problem gives rise, via (7), to a matrix $P(t)$ satisfying (1)–(6). (C): Conversely, if $P(t)$ satisfies (1)–(6), there exists a unique ψ, positive and supported by $x\geqq0$, such that $P(t)$ is given by (7). This is proved by considering the semi-group $\{T_t\}$, on the Hilbert space $L_2(\pi)$ of sequences $\{f(n)\}$ with scalar product $(f,g)=\sum\pi_nf(n)\overline{g(n)}$, defined by $(T_tf)(i)=\sum_j P_{ij}(t)f(j)$. It is shown that T_t is symmetric and positive definite, with $\|T_t\|\leqq1$. Putting $f(i)=\delta_{i0}$, it follows readily that $P_{00}(t)=(T_tf,f)$ is completely monotonic, hence representable as $\int e^{-xt}d\psi(x)$; the general representation (7) can be deduced by repeated differentiation, using (1) and (2). (D): The preceding results, together with some rather intricate arguments which cannot be summarized here, lead to a uniqueness criterion (stated below only for $\mu_0=0$). This asserts the equivalence of the three statements (i) there is only one $P(t)$ satisfying (1)–(3) and $|\sum_{j=0}^n P_{ij}(t)|\leqq M$, (ii) the Stieltjes moment problem has a unique solution, (iii) the series $\sum(\pi_n+(\lambda_n\pi_n)^{-1})$diverges. It is also shown that the "minimal" solution of (1)–(6) is "honest", i.e., has row sums unity, if and only if $\sum(\lambda_n\pi_n)^{-1}(\pi_0+\cdots+\pi_n)$ diverges. (E): Further interesting results relate to the case of non-uniqueness when $\sum(\pi_n+(\lambda_n\pi_n)^{-1})$ diverges, to total positivity properties of $P(t)$ and to its sign-variation diminishing properties when operating in the sequence spaces l and m. *G. E. H. Reuter* (Manchester).

Broadbent, S. R.; and Hammersley, J. M. Percolation processes. I. Crystals and mazes. Proc. Cambridge Philos. Soc. 53 (1957), 629–641.

Hammersley, J. M. Percolation processes. II. The connective constant. Proc. Cambridge Philos. Soc. 53 (1957), 642–645.

If one has a fluid moving through a medium according to some stochastic law, then the authors would call the process a diffusion process if the random property is associated with the fluid, but a percolation process if the random property is associated with the medium. Most of each paper deals specifically with walks over sets of points for which the possible paths between points are stochastic. The authors state "The present papers are a preliminary exploration of percolation process; and, although our conclusions are somewhat scanty, we hope we may encourage others to investigate this terrain...". The papers contain 13 examples and 8 theorems. Part II contains the proof of one of the theorems in part I. *G. Newell* (Providence, R.I.).

Longuet-Higgins, M. S. Statistical properties of an isotropic random surface. Philos. Trans. Roy. Soc. London. Ser. A. 250 (1957), 157–174.

Some of the statistical properties of a random, moving surface, having a correlation function of general form, were found by the author in a previous paper [same Trans. **249** (1957), 321–387; MR **19**, 328]. The author here specializes these results to the case in which the surface is isotropic; i.e., its statistical properties are independent of direction. In this way he obtains, for example, the distributions of elevation and gradient, the mean number of zeros along a line in an arbitrary direction, the average density of maxima and minima per unit area, the distribution of the velocities of zeros, etc. *H. A. Hauptman.*

Skellam, J. G.; and Shenton, L. R. Distributions associated with random walk and recurrent events. J. Roy. Statist. Soc. Ser. B. **19** (1957), 64–111 (discussion 111–118).

Various distribution questions concerning the renewal process, such as the forward and backward delays and the census distribution are studied by manipulations of generating functions and Laplace transforms. Many results are expressed in terms of contour integrals. Graphs and tabulated examples are given. The discussion contains historical remarks and different points of view as to the relative importance of "theorems" vs. "formulas". The rigorous derivation of the delay distributions, also under other assumptions, had been given by Dynkin [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 247–266; MR **17**, 865]. *K. L. Chung* (Syracuse, N.Y.).

Beneš, V. E. On queues with Poisson arrivals. Ann. Math. Statist. **28** (1957), 670–677.

"The system studied consists of a service unit and a queue of customers waiting to be served. Service-times of customers are independent, nonnegative variates with the common distribution $B(v)$ having a finite first moment b_1. Customers arrive in a Poisson process of intensity λ; they form a queue and are served in order of arrival, with no defections from the queue.

"Let $W(t)$ be the time a customer would have to wait if he arrived at t. The forward Kolmogorov equation for the distribution of $W(t)$ is solved in principle by the use of Laplace integrals, and $E\{\exp\{-sW(t)\}\}$ is determined in terms of $W(0)$ and the root of a possibly transcendental equation. It is shown that any analytic function of the root can be expanded in Lagrange's series, which provides a way of actually computing the transition probabilities of the process. Let z be the first zero of $W(t)$. A series for $E\{\exp\{-rz\}\}$ is obtained, and it is proved that $\mathrm{pr}\{z<\infty\}=1$ if and only if $\lambda b_1 \leq 1$. The covariance function R of $W(t)$ is determined. If $B(v)$ has a finite third moment, then R is absolutely integrable, and the spectral distribution of $W(t)$ is absolutely continuous." [From the author's summary.] *D. V. Lindley* (Cambridge, England).

Akaike, Hirotugu. On ergodic property of a tandem type queueing process. Ann. Inst. Statist. Math., Tokyo **9** (1957), 13–21.

The output of a $GI/G/1$ queue becomes the input of a $-/G/1$ queue, with the proviso that the second phase begins for customer j only when customer $(j+1)$ begins service at the first phase. (This is so that certain Markov properties are obtained.) It is shown that a necessary and sufficient condition for ergodicity of the system, when the mean inter-arrival time μ is greater than the mean service time at the first phase, is that μ also be greater than the mean service time at the second phase. The extension to several phases is indicated. *D. V. Lindley.*

★ **Яглом, А. М.; и Яглом, И. М. [Yaglom, A. M.; and Yaglom, I. M.] Вероятность и информация. [Probability and information.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 160 pp. 2.70 rubles.

A semi-popular account in four chapters: probability, entropy and information, solution of certain logical problems by information, application of information to communication. There are 2 appendices: properties of convex functions, some inequalities.

See also: Measure, Integration: Krickeberg. **Harmonic Functions, Convex Functions:** Hunt. **Optics, Electromagnetic Theory, Circuits:** Faure and Savelli.

STATISTICS

★ **Czechowski, T.; Olekiewicz, M.; Perkal, J.; and Wiśniewski, W. L., Editors. Statystyka jako metoda poznawcza. [Statistics as a method of research.]** Polskie Towarzystwo Przyrodników im. Kopernika. Państwowe Wydawnictwo Naukowe, Warsaw, 1956. 238 pp. zł 15.10.

Proceedings of a conference arranged by the Copernican Society of Polish Naturalists. The volume contains seven papers, by M. Fisz, W. Sadowski, T. Czechowski, J. Perkal, M. Lacki, M. Kacprzak and M. Olekiewicz, and a record of the discussion. The central points of the discussion are the role of statistics in scientific research and the desirable form of cooperation between the naturalist and the statistician. *J. Neyman* (Berkeley, Calif.).

★ **DuBois, Philip H. Multivariate correlational analysis.** Harper & Brothers, New York, 1957. xv+202 pp. (2 plates) $4.50.

The book seems to be intended primarily for social scientists and provides systematic procedures and charts for the computation of multiple, partial, and part correlation coefficients, as well as the coefficients of linear regressions. Procedures for the computations of factor analysis are given and related to the computations for partial correlation. The two sentences following equation (31) on page 91 need to be combined and corrected.

S. Kullback (Washington, D.C.).

Andreoli, Giulio. Medie e loro processi iterativi. Giorn. Mat. Battaglini (5) **5**(85) (1957), 52–79.

Sono dapprima richiamati, sommariamente, alcuni concetti elementari relativi alle definizioni delle medie, limitandosi quasi esclusivamente, per semplicità, al caso di due quantità. Semplici dimostrazioni sono date per tipi di medie cosiddette simmetriche, od omogenee, o traslative. Proprietà fondamentali relative all'applicazione, alternativamente ripetuta, di due diverse definizioni di media. Generalizzazioni al caso di più di due quantità e di più di due definizioni di media. Esempi vari. *T. Viola.*

Andreoli, Giulio. Aspetto gruppale e funzionale delle medie. Giorn. Mat. Battaglini (5) **5**(85) (1957), 12–30.

L'Autore espone, in via discorsiva, alcune idee generali sul concetto di media. Vengono definite le "trasformazioni delle medie", e quindi considerate particolari classi di medie fra loro associate per certe prefissate relazioni funzionali. Sono esaminate le medie dal punto di vista di seriazioni formate con gli stessi valori numerici, permutati in quanto all'ordine, oppure con gruppi di essi sostituiti da loro medie parziali. Infine è accennato lo studio delle medie deducibili direttamente da quella aritmetica e le loro relazioni con certe teorie matematiche.

T. Viola (Roma).

van Dantzig, D. The development of mathematical statistics during the last ten years. Statistica, Neerlandica 9 (1955), 233–242. (Dutch. English summary)

Zinger, A. A. On independence of polynomial and quasi-polynomial statistics. Dokl. Akad. Nauk SSSR (N.S.) 110 (1956), 319–322. (Russian)

Let X be an n-dimensional random vector with independently distributed components, and let P_k and Q_h be polynomial statistics in the components of degree k and h respectively, involving each component essentially in degree k and h. (1) Then if P and Q are independent, the components of X have finite moments of all orders. (2) If Q is linear, then $E(\exp(h|x_j|))<\infty$ for all h; i.e., the characteristic functions of all components are entire functions. (3) Furthermore, $E(\exp(h|x_j|))=O(\exp(Bh^A))$, where $B>0$, $A>1$ are constants and $0<h<\infty$. (4) If P and Q are independent, and if the characteristic function of the component x_j vanishes for no t, then x_j is normally distributed. The proof depends on the following lemma: Let z be a non-negative random variable with distribution function $F(z)$; if for some α $(0<\alpha<1)$ the integral $\int_0^\infty (1-F(\alpha z))^{-1}dF(z)$ is finite, z has finite moments of all orders $P>0$. A statistic $S(X)$ is called quasi-polynomial if there exist a continuous function $g(S)$ and two non-negative polynomials $R(x)$ and $r(x)$ of degree $\geqq 1$ such that, for all X, $R(X)\geqq g(S(X))\geqq r(X)\geqq 0$. Let S_m be a quasi-polynomial statistic such that R and r are both of degree m, and r involves all components essentially in degree m; then if P_k is an S_k and Q_h is an S_h, all the above theorems hold for P_k and Q_h. *A. A. Brown.*

Aoyama, Hirojiro. Sampling fluctuations of the test reliability. Ann. Inst. Statist. Math., Tokyo 8 (1957), 129–143.

The author derives expectation and variance formulas for the reliability measures of Kuder-Richardson and Spearman-Brown, assuming (i) sampling from a population of examinees, the test items being fixed, and (ii) vice versa. The formulas are asymptotic, or give upper bounds; they are rather complicated. *G. Elfving.*

White, John S. Approximate moments for the serial correlation coefficient. Ann. Math. Statist. 28 (1957), 798–802.

Formulas are derived for the moments of the distribution defined by the density function

$$f(t)=C(1-2\rho t+\rho^2)^{-N/2}(1-t^2)^{(N-1)/2}$$

in $-1\leqq t\leqq 1$. The distribution in question is an approximation to that of the serial correlation coefficient of a circular first order Gaussian auto-regressive process. *G. Elfving* (Helsingfors).

Somerville, Paul N. Optimum sampling in binomial populations. J. Amer. Statist. Assoc. 52 (1957), 494–502.

When a decision must be made to "adopt" one of two binomial populations with unknown frequency parameters θ_0 and θ_1, the cost attributable to a wrong decision can often be approximated by $K|\theta_0-\theta_1|$. The total cost of taking a sample of size n from both populations is often c_0+c_1n. This paper explores the minimax value of n and the total minimax loss, especially when $c_1\ll K$, in which case the quantities are asymptotically $0.1534(K/c_1)^{2/3}$ and $0.4603(K^2/c_1)^{1/3}+c_0$. *L. J. Savage* (Chicago, Ill.).

Blumen, Isadore. On the ranking problem. Psychometrika 22 (1957), 17–27.

A judge starts with a set of n objects in an arbitrary order. He selects, at random, an adjacent pair and, with probability p, ranks these in accordance with a standard permutation of the n objects. From the resulting new order he again selects an adjacent pair, etc. For this Markov process the stationary distributions are found as a function of p. For these distributions a sufficient, for p, statistic s together with its distribution is given. From s an unbiased estimator $\hat{p}$ of p is computed. It is stated that, as n increases $\hat{p}$ becomes asymptotically normal with a given variance. *C. H. Kraft* (East Lansing, Mich.).

Trybula, S. On a problem of prognosis. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 859–862, LXXIV. (Russian summary)

The minimax squared error predictor of one binomial variable from another, given as an example by Hodges and Lehmann [Ann. Math. Statist. 21 (1950), 182–197; MR 12, 36], is extended to the multinomial case. *J. L. Hodges, Jr.* (Berkeley, Calif.).

Ramachandran, K. V.; and Khatri, C. G. On a decision procedure based on the Tukey statistic. Ann. Math. Statist. 28 (1957), 802–806.

Let $\{x_{ij}\}$ $(i=1, 2, \cdots, k;\ j=1, 2, \cdots, n)$ be k independent samples of size n from the normal populations $N(\mu_i, \sigma_i^2)$. Let $\bar{x}_i=\sum_{j=1}^n (x_{ij})/n$, $\bar{x}=\sum_{i=1}^k \bar{x}_i/k$, $(nk-1)s_0^2=\sum_{i=1}^k \sum_{j=1}^n (x_{ij}-\bar{x})^2$, $\bar{x}_{\max}=\max(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$, $\bar{x}_{\min}=\min(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$. Let D_{00} denote the decision that the k means are equal, and let D_{ij} $(i\neq j;\ i, j=1, 2, \cdots, k)$ denote the decision that D_{00} is incorrect, and $\mu_i=\mu_{\min}$ and $\mu_j=\mu_{\max}$. The pair (μ_i, μ_j) is said to have slipped by the amount $\Delta(>0)$ if $\mu_i=\mu-\Delta$ and $\mu_j=\mu+\Delta$, and all other μ_h $(h\neq i, j)$ are equal to some μ. The problem is to find a statistical procedure for selecting one of the decisions (D_{00}, D_{ij}) $(i\neq j;\ i, j=1, 2, \cdots, k)$ which, subject to certain restrictions (a), (b), and (c), will maximize the probability of making the correct decision when one of the pairs has slipped. The answer is: if $\bar{x}_i=\bar{x}_{\min}$, $\bar{x}_j=\bar{x}_{\max}$, select D_{ij} or D_{00} when $n(\bar{x}_j-\bar{x}_i)/\{(nk-1)s_0^2\}^{\frac{1}{2}}>q_\alpha$ or $\leqq q_\alpha$ respectively, where q_α is a constant whose value is determined by the restriction (a). This method is similar to that used by Paulson and Traux, and the result is a justification for the statistical procedure based on the Tukey Studentized range.

The restrictions are: (a) when all the means are equal, D_{00} should be selected with any assigned probability $1-\alpha$; (b) the decision procedure is invariant under location and scale transformations of the variate; (c) the decision is symmetric for all pairs (i, j), in the sense that the probability of making the correct decision must be the same for the same amount of slippage of any pair (i, j). *T. Kitagawa* (Fukuoka).

Tukey, John W. Antithesis or regression? Proc. Cambridge Philos. Soc. 53 (1957), 923–924.

The "antithetic variate method" of Hammersley and Morton [same Proc. 52 (1956), 449–475; MR 18, 336] for sampling in Monte Carlo is a special case of a regression estimate already known in the theory of sampling. *A. S. Householder* (Oak Ridge, Tenn.).

Durbin, J. Testing for serial correlation in systems of simultaneous regression equations. Biometrika 44 (1957), 370–377.

The results previously derived by Durbin and Watson

for single equation models [Biometrika 37 (1950), 409–428; 38 (1951), 159–178; MR 12, 512; 13, 144] are extended to models with $p+1$ interrelated equations in $p+1$ dependent or endogenous (y) and $k+q$ ($q \geq p$ and $k \geq 0$) independent or exogenous (x) variables. The equations are of the form $Ay=Bx+\varepsilon$, where ε represents the vector of errors or disturbances. It is assumed that n simultaneous observations are obtained on the variables. Results are derived to test for serial dependence in a given equation of the system, assuming that q of the elements of the corresponding vector of B are zero.

For a just-identified equation in the system ($q=p$), Durbin uses the same statistic as in the Durbin-Watson articles, with $k'=k+p-1$. The statistic is

$$d=\sum_{t=2}^{n}(z_t-z_{t-1})^2/\sum_{t=1}^{n}z_t^2,$$

where the z's are the sample residuals for the equation being studied. As before, it is assumed that one of the exogenous variables represents the intercept; hence, it has a unit vector in B. Reduced form estimation is used.

A similar statistic (with $k'=k+q-1$) can be used for over-identified equations ($q>p$).

These results are more general than is implied by the author. He implies that the number of endogenous variables (H) in the equation under study is $p+1$; his results hold if $H<p+1$. Also $k+q$ can be replaced by arbitrary K; in this case the equation is just-identified when $H-1=K^{**}$, where K^{**} is the number of exogenous variables in the system but not in the given equation.

R. L. Anderson (London).

★ **Anderson, T. W. An introduction to multivariate statistical analysis.** Wiley Publications in Statistics. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1958. xii+374 pp. $12.50.

The author considers only statistical analysis based on the multivariate normal distribution. In the introductory chapter 1, he classifies the statistical methods of multivariate analysis into five categories: (1) Correlation; (2) Analogues of univariate statistical methods; (3) Problems of coordinate systems; (4) More detailed problems; and (5) Dependent observations. In Chapter 2, the multivariate normal distribution is defined and the important related properties considered. Maximum-likelihood estimates of the mean vector and covariance matrix are considered in Chapter 3, as well as the distribution of the sample average vector. In Chapter 4, the various sample correlation coefficients are studied. The generalized T^2-statistic (Hotelling's generalization of Student's t-statistic) forms the subject matter of Chapter 5. Chapter 6, on the classification of observations, considers a general problem of statistical inference, with particular consideration of observations from multivariate normal populations. The Wishart distribution, the generalized variance, and related topics are the subject of Chapter 7. Chapter 8, on the general linear hypothesis, generalizes the univariate least squares theory, regression analysis, and the analysis of variance. Tests of the mutual independence of sets of variables are studied in Chapter 9. Problems related to the homogeneity (equality) of covariance matrices and mean vectors are examined in Chapter 10. Principal components, linear combinations of random variables which have special variance properties, are studied in Chapter 11. Canonical correlations, the correlations between linear combinations of sets of variables, are the subject of Chapter 12. The distributions of certain characteristic roots (roots of determinantal equations) which appear in many invariant tests (and related characteristic vectors) are considered in Chapter 13. Chapter 14 offers a brief look at some more advanced topics of multivariate analysis. An appendix of 19 pages on matrix theory and a bibliography of 17 pages are also included. Chapters 2, 3, and 4 may be classified as in the first of the above mentioned categories; chapters 5, 6, 7, 8, and 10 as in the second; chapters 11, 12, and 13 as in the third; chapter 9 as in the fourth; and chapter 14 very briefly mentions some problems in various of the above mentioned categories. It seems hardly necessary to say that matrix algebra is fully exploited.

The author has admirably achieved his objective of producing a book that will not only serve as an introduction to many topics in multivariate analysis to students, but will also be used as a reference by other statisticians. In referring to the vector coefficient of alienation and the square of the vector correlation coefficient, the author says on page 245, "Of course, measures such as these are not completely satisfactory as generalizations because they omit information relating the two sets; that is, the relationship between $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ cannot be expressed in a single number." However, the concepts of information theory do yield a measure of the information in one random vector about another random vector, which, for the case in question, is expressible in terms of the canonical correlations. The format and typography of the book are very good. There are certain inconsistencies in the exposition which might offend a purist. There are a few misprints. This book is a valuable addition to the literature of mathematical statistics.

S. Kullback (Washington, D.C.).

Gjeddebaek, N. F. Contribution to the study of grouped observations. II. Loss of information caused by grouping of normally distributed observations. Skand. Aktuarietidskr. 39 (1956), 154–159 (1957).

[For part I see Skand. Aktuarietidskr. 32 (1949), 135–159; MR 11, 446.] When a considerable amount of data is available, it is customary to group it into intervals before estimating the mean and standard deviation. To derive an upper bound for the asymptotic efficiency of this method of using grouped data when the probability distribution is normal, the author derives the asymptotic efficiency of using the maximum-likelihood estimates based on the limited information of the number of observations which fall in each cell. Computations are carried out for the cases in which the intervals have equal width, and in which the mean lies (a) in the center, (b) at an edge, or (c) one-quarter of the way toward an edge of one of these intervals. These results indicate that at least the maximum-likelihood estimates are quite efficient if the interval width is less than two standard deviations. Of the three cases above, the worst seems to be (b).

H. Chernoff (Stanford, Calif.).

Dixon, W. J. Estimates of the mean and standard deviation of a normal population. Ann. Math. Statist. 28 (1957), 806–809.

Several simple estimates of the mean and standard deviation of a normal population are discussed. The efficiencies of these estimates are compared to the sample mean and sample standard deviation and to the best linear unbiased estimates. Little efficiency is lost when simple rather than optimum weights are used. [From the author's summary.] *H. Chernoff.*

Gutman, M. On estimating an integral from the square of the density. Zastos. Mat. 3 (1958), 329–336. (Polish. Russian and English summaries)

The author discusses the problem of estimating $\int_{-\infty}^{\infty} f^2(x)dx$ where $f(x)$ is the probability density of the normal distribution. His result is that if S is defined by $ns^2=\sum_{i=1}^{n} (x_i-\bar{x})^2$, where $x_1, x_2, \cdots, x_n$ is an n-element sample drawn from a normal general population, then the estimator

$$(2\pi)^{-\frac{1}{2}} S^{-1}\,\Gamma(\tfrac{1}{2}(n-1))/\Gamma(\tfrac{1}{2}(n-2))$$

is an unbiased and asymptotically most effective estimator of the above integral.

The theorems contained in the paper imply also that the estimator $(n-3)/ns^2$ is an unbiased estimator of the precision square h^2 of the normal distribution.

From the author's summary.

Watson, G. S. Sufficient statistics, similar regions and distribution-free tests. J. Roy. Statist. Soc. Ser. B. 19 (1957), 262–267.

An expository paper. *E. L. Lehmann.*

Gihman, I. I. Über ein nichtparametrisches Kriterium der Homogenität der k-Stichproben. Teor. Veroyatnost. i Primenen. 2 (1957), 380–384. (Russian. German summary)

In order to test the hypothesis H_0 that N independent random variables, grouped into k samples of sizes $n_1, \cdots, n_k$, have a common continuous distribution function (d.f.), the author considers the statistic

$$D_k{}^2=\max_{-\infty<x<\infty} \sum_{i=1}^{k} n_i \{\hat{F}_i(x)-\hat{F}_0(x)\}^2,$$

where $\hat{F}_i$ is the empirical d.f. of the ith sample ($i=1, \cdots, k$), and $\hat{F}_0=\sum_{i=1}^{k} n_i F_i/N$. For $k=2$, this is a statistic proposed by N. V. Smirnov [Bull. Math. Univ. Moscou 2 (1939), no. 2; MR 1, 345]. When H_0 is true, the limiting distribution, for $\min(n_1, \cdots, n_k)\to\infty$, of $D_k{}^2$ is shown to be identical with that of (*) $\max_{0\leq t\leq 1}\sum_{i=1}^{k-1} \xi_i^2(t)$, where $\xi_i(t)$, $0\leqq t\leqq 1$, is a (separable) Gaussian process with mean 0 and covariance $E\xi_i(s)\xi_i(t)=s(1-t)$, $0\leqq s\leqq t\leqq 1$, and the random functions $\xi_1, \cdots, \xi_k$ are mutually independent. An explicit expression for the d.f. of the random variable (*) in the form of an infinite series is obtained.

W. Hoeffding (Chapel Hill, N.C.).

Fraser, D. A. S. On the combining of interblock and intrablock estimates. Ann. Math. Statist. 28 (1957), 814–816.

Let $x=(x_1, \cdots, x_r)$ and $y=(y_1, \cdots, y_r)$ be independent estimates of the parameter $\eta=(\eta_1, \cdots, \eta_r)$ having nonsingular covariance matrices V and W respectively. If x and y follow multivariate normal distributions, then $(xV^{-1}+yW^{-1})(V^{-1}+W^{-1})^{-1}$: (i) is a sufficient statistic for η; (ii) is complete; (iii) is unique and is an unbiased estimate of η; and (iv) has minimum concentration ellipsoid among all unbiased estimates. The author applies the result to the combining of intra- and inter-block estimates for treatment contrasts with respect to incomplete block designs, and shows that weighting the estimates by reciprocal variances is appropriate only when VW^{-1} is a diagonal matrix. *M. Zelen.*

★ **Hendricks, Walter A. The mathematical theory of sampling.** The Scarecrow Press, New Brunswick, N. J., 1956. vii+364 pp. $7.50.

See also: Numerical Methods: Das. **Probability:** Steck. **Statistical Thermodynamics and Mechanics:** Lebowitz and Frisch. **Biology and Sociology:** Tryon; Gerard.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

★ **Veiga de Oliveira, Fernando Vasco Alves da. Soluções homográficas no problema generalizado dos n corpos. [Homographic solutions to the generalized problem of n bodies.]** Faculdade de Ciências, Lisboa, 1956. 82 pp.

The generalized n-body problem is the problem of determining the motion of a system consisting of n particles subject to internal forces derived from a potential energy function which is a homogeneous function of the coordinates of the particles. A homographic solution of the problem is a solution such that the figure formed by the n particles remains similar to the figure existing at the initial instant. Certain particular homographic solutions, e.g., Lagrange's equilateral triangle solution of the three-body problem, have long been known in celestial mechanics.

In the present monograph, the author gives a general discussion of the existence and elementary properties of homographic solutions. The results include and unify the previously known results, and extend them in various directions. It soon becomes necessary to introduce assumptions concerning the forces which are more specific than those indicated above. In much of the work it is assumed that the forces are attractive or repulsive forces acting between the particles, and that the force acting between two particles is proportional to a power of their mutual distance, where the power is the same for all pairs of particles. It is found that the case in which the forces are inversely proportional to the cubes of the distances is exceptional in many respects. The latter part of the monograph is devoted to an elaborate study of the first-order stability of homographic solutions. As the known results for the classical particular cases suggest, the stability of a homographic solution generally depends in an intricate way on the masses of the particles and other parameters.

The author uses, without much in the way of explanation, certain highly individual methods and notations which he seems to have expounded previously in other papers. For this reason, many of the details of the arguments are difficult to follow. However, it appears that the principal conclusions are correct. A few typographical errors have been encountered, but none which cause serious trouble. *L. A. MacColl* (New York, N.Y.).

Maravall Casesnoves, Dario. The quadratic complex of lines of constant moment and other problems of classical mechanics. Gac. Mat., Madrid (1) 9 (1957), 6–13. (Spanish)

This paper consists of three unrelated short notes, as follows. (1) Given a system of sliding vectors in 3-space, the lines with respect to which the system has a moment with a prescribed value k form, as is shown, a quadratic complex. The author studies the fundamental properties of this complex. The differential equation of the curves of

the complex is derived, and is solved by quadratures in certain cases. (2) Considering a curve of pursuit in the usual planar case, the author gives a simple construction for finding the center of curvature of the curve at a typical point. (3) The linear differential equations for small sliding motions of a heavy chain on a smooth curve are derived, and the stability of configurations of equilibrium of the system is investigated. *L. A. MacColl.*

Bottema, Oene. First integrals of dynamical systems. Z. Angew. Math. Phys. **8** (1957), 418–420.

If S is a set of n Lagrangian equations with $m \leq n$ known mutually independent first integrals $F_i(q, \dot{q}) = c_i$, the author shows by two examples that a solution of a system S' consisting of the m first integrals and $n-m$ of the equations S is not necessarily also a solution of the set S. *E. B. Schieldrop* (Oslo).

Creech, M. D. The mobility of levers having uniformly distributed mass. J. Appl. Mech. **24** (1957), 475–477.

Consider a uniform rigid lever having a total mass M and a length L and having mobilities z_i attached at positions α, measured from, say, the left-hand end. The author computes the mobility looking into the left end by dividing the lever into n equal blocks, finding an expression for the mobility of such a finite set of sections, and passing to the limit as $n \to \infty$. It would appear that the problem could be solved more readily by treating the lever as a single rigid entity possessing linear and rotary kinetic energies rather than as an infinite sequence of short levers rigidly joined. *H. M. Trent.*

Novoselov, V. S. Reduction of a problem of non-holonomic mechanics to a conditional problem of the mechanics of holonomic systems. Leningrad. Gos. Univ. Uč. Zap. **217**. Ser. Mat. Nauk **31** (1957), 28–49. (Russian)

This work represents an interesting generalization of the equations of the motion of a material system, if the non-holonomic constraints are not linear with respect to velocities. The non-holonomic constraints have the form $F_k(q_i, \dot{q}_i, t) = 0$ and the variations δq_i are defined so as to satisfy the equations $\sum_{i=1}^{s} \frac{\partial F_k}{\partial \dot{q}_i} \delta q_i = 0$. Then the equation of the motion can be written in the usual Lagrangian form. The expressions for the generalized reaction-forces are obtained. The equations of the motion can be reduced to canonical form. If we know two integrals, which are in involution with the generalized forces of reactions, then an integral can be obtained by forming the Poisson's brackets. *P. Musen.*

Lévi, Robert. Dispositif mécanique résolvant certains problèmes de recherche opérationnelle. C. R. Acad. Sci. Paris **245** (1957), 2166–2168.

Pour rendre stationnaire une fonction linéaire d'inconnues qui sont elles-mêmes astreintes à certaines relations linéaires à coefficients tous égaux et qui sont nécessairement positives ou nulles, un dispositif mécanique analogique permet de fournir une solution abrégée sous la forme d'un tableau de relations d'où la solution numérique du problème se déduit immédiatement.

Résumé de l'auteur.

Guichardet, Alain. Quelques propriétés des prévisions en mécanique. C. R. Acad. Sci. Paris **245** (1957), 2211–2213.

Remarques sur les lois de probabilités associées aux grandeurs physiques et aux états de la Mécanique quantique et sur la précision des mesures physiques.

Résumé de l'auteur.

Winogradzki, Judith. Sur les retournements de l'espace, du temps et de l'univers dans la théorie des spineurs de Dirac. II. Représentations normales particulières. Les six représentations fondamentales. C. R. Acad. Sci. Paris **245** (1957), 2206–2208.

Etude de représentations spinorielles normales particulières, définies à partir de la loi de transformation des semi-spineurs pour les retournements de l'espace, du temps et de l'Univers. *Résumé de l'auteur.*

Ovchinnikov, P. F. The differential equation of lateral vibrations in a revolving bar of variable cross-section taking into account the shear and gyroscopic moment. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 147–154. (Ukrainian. Russian and English summaries)

Egorov, V. A. Some problems of dynamics of flight to the moon. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 46–49. (Russian)

The problem of flight to the moon, taking into account only the principal forces acting on a rocket, reduces to the circular restricted problem of three bodies. The author claims that during the years 1953–1955, he investigated systematically the planar case of the circular restricted three-body problem at the Institute for Mathematics of the Academy of Sciences of the USSR. In the paper under review, he discusses the problem of hitting the moon, the problem of circumnavigating the moon, the problem of periodic flights, classification of trajectories, and so on. No proofs are given. *E. Leimanis* (Vancouver, B.C.).

See also: Classical Thermodynamics, Heat Transfer: Kapur; Tawakley.

Statistical Thermodynamics and Mechanics

Bazarov, I. P. Statistical theory of systems of charged particles with account of short range forces of repulsion. Soviet Physics. JETP **5** (1957), 946–952.

The author modifies the usual Debye-Huckel results, valid for large distances, for the pair-distribution function of a classical plasma, by the introduction of short range hard sphere repulsive forces. This is done in two steps. First the Coulomb potential is replaced by a potential of the form $\Phi^{(0)}(r) = \begin{cases} 0, & r < \sigma \\ e_a e_b / r, & r > \sigma \end{cases}$. The resultant distribution function, calculated using the standard Debye-Huckel approximations, now has no singularity at the origin. It is possible from here to use perturbation theory on the Mayer f-functions, by means of methods due to Bogoljubov, to include the first approximation to the free energy for an additional repulsive sphere potential $\Phi^{(1)}(r) = \begin{cases} \infty, & r < \sigma \\ 1, & r > \sigma \end{cases}$. *J. C. Ward* (Coral Gables, Fla.).

Lebowitz, Joel L.; and Frisch, Harry L. Model of non-equilibrium ensemble: Knudsen gas. Phys. Rev. (2) **107** (1957), 917–923.

Gas molecules are assumed to have impulsive collisions with the walls of their container but not with each other (Knudsen gas), with the velocities before and after collision stochastically determined by time independent

transition probabilities. After stating that "without any loss in generality we may consider our container to be a right cylinder...", for which no justification is apparent, the authors proceed to treat the equivalent of a one-dimensional gas or a gas between two parallel infinite plates. The time evolution of the system is described, and it is shown that the system approaches a stationary state, even if the walls have different temperatures. The stationary state is found explicitly in a special case.

G. Newell (Providence, R.I.).

Thouless, D. J. Use of field theory techniques in quantum statistical mechanics. Phys. Rev. (2) **107** (1957), 1162–1163.

In order to find a statistical description of a fermion gas, Matsubara [Progr. Theoret. Phys. **14** (1955), 351–378; MR **17**, 695] decomposed the field $\psi(x)$ into two parts, $\psi=\psi_+ + \psi_-$, which do not anticommute. A normal product was defined by the ordering $(\psi_-{}^*)(\psi_-)(\psi_+{}^*)(\psi_+)$, and the decomposition was specified by requiring $\langle N[\psi^*(x)\psi(x')]\rangle_{\mathrm{av}}$ to vanish in the equilibrium state at the given temperature. He then showed that $\langle N[\psi^*(x)\psi(x)\psi^*(x')\psi(x')]\rangle$ does not vanish exactly, but tends to zero as $1/V$ when the size of the system increases. {His expression (3.21) is incomplete, but this does not affect his conclusion.} The present author defines the normal product by the ordering $(\psi_+{}^*)(\psi_-)(\psi_-{}^*)(\psi_+)$, and finds that its average is exactly zero. He concludes that Matsubara's $1/V$ terms are spurious. The boson field is also treated.

N. G. van Kampen (Washington, D.C.).

See also: Partial Differential Equations: Barenblatt. **Structure of Matter:** Goldstein. **Fluid Mechanics, Acoustics:** Lal and Bhatnagar. **Optics, Electromagnetic Theory, Circuits:** van Kampen. **Quantum Mechanics:** Magalinskii and Terletskii.

Elasticity, Plasticity

Smith, G. F.; and Rivlin, R. S. Stress-deformation relations for anisotropic solids. Arch. Rational Mech. Anal. **1** (1957), 107–112.

The solids considered are such that the components of true stress are polynomial functions of the displacement gradients, though not necessarily derivable from a strain-energy. These functions are restricted by an invariance requrement under rigid-body rotation. Further restrictions arising from symmetry properties of the material are studied here. Calculations are carried out for the monoclinic and rhombic crystal classes, but the method is general.

R. Hill (Nottingham).

Truesdell, Clifford. L'ipoelasticità. Confer. Sem. Mat. Univ. Bari no. **29** (1957), 16 pp. (1 photograph).

This paper describes the theory of hypoelasticity. It is an excellent introduction to the subject for any mathematician who can read Italian.

D. R. Bland.

Wigglesworth, L. A. Stress distribution in a notched plate. Mathematika **4** (1957), 76–96.

The distribution of stress in a semi-infinite elastic plate, with a thin notch perpendicular to its straight edge, is found for a state of plane strain or generalized plane stress. The general problem is reduced, by using known solutions for the complete semi-infinite plate, to that of the notched plate under tractions applied only to the edges of the notch. The reduced problem requires the solution of a singular integral equation, and an exact formal solution is found by means of a Mellin transformation. A result of high numerical accuracy is obtained from this solution for the case of a uniform tension parallel to the edge of the plate.

W. R. Dean.

Eras, Gunter. Eine Anwendung komplexer Spannungsfunktionen in der Plattentheorie. Wiss. Z. Tech. Hochsch. Dresden **6** (1956/57), 685–690.

The bending of an elastic circular plate with a concentric hole, supported at four points of the inner edge and under the action of an axisymmetrical, otherwise arbitrary, transverse load is studied with complex variables. The solution is built up by superposition of an elementary part which depends upon the specific load distribution, and a nonsymmetrical part which is independent of it. The two complex functions governing the problem are taken in the form of power series with coefficients that can be determined at the boundary, where the edge reactions are developed into a Fourier series. Thus, the method corresponds in its nature to the one used in the case of an eccentric force acting on a circular plate [see, e.g., A. Clebsch, Theorie der Elasticität fester Körper, Teubner, Leipzig, 1862, p. 319 ff.].

W. Schumann.

Szelągowski, Franciszek. One-directional tension of a circular ring. Rozprawy Inż. **4** (1956), 431–439. (Polish. Russian and English summaries)

The state of stress in a flat circular elastic ring with rigid inclusion under uniaxial tension is established by using Laurent series.

A. M. Freudenthal.

Neou, Ching-Yuan. A direct method for determining Airy polynomial stress functions. J. Appl. Mech. **24** (1957), 387–390.

Two dimensional stress problems in isotropic bodies may be reduced to the determination of an Airy stress function Φ satisfying the biharmonic equation of compatibility and certain boundary conditions. This paper shows that, for continuously loaded rectangular strips, the direct method of expressing Φ in the form of a doubly-infinite power series in the cartesian coordinates (x, y), given by $\Phi=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{m,n}x^m y^n$ may be used, and the coefficients $C_{m,n}$ may be determined from the equation of compatibility and the boundary conditions only. Two particular examples are given.

R. M. Morris.

Mahovikov, V. I. On approximate conformal representations and their application to the theory of elasticity. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 20–37. (Ukrainian. Russian and English summaries)

Transformation of the Cristoffel-Schwarz integral is proposed in this paper. The integral is written as an irrational function plus an integral of the Cristoffel-Schwarz type with power indexes of higher degree. This allows us to represent regions by means of comparatively simple functions.

Methods are proposed for the approximate conformal representation of uni-connected and multi-connected regions. These methods can be applied to the solution of the two-dimensional anisotropic problem of the theory of elasticity.

Author's summary.

Belova, V. I. Distribution of strain in a stretched plane plate with axi-symmetric inset. Leningrad. Gos. Univ. Uč. Zap. **217.** Ser. Mat. Nauk **31** (1957), 236–253. (Russian)

Savruk, M. A. Influence of a circular opening on the stress in a flexible half-plane. L'vov. Politehn. Inst. Nauč. Zap. **30**, Ser. Fiz.-Mat. No. 1 (1955), 65–71. (Russian)

Finzi, Leo. Legame fra equilibrio e congruenza e suo significato fisico. I. Atti Accad. Naz. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 205–211.

The author establishes an analogy between the general integrals of the equilibrium equations (Navier's equations) for a continuous body in absence of body forces, and the compatibility equations, in terms of strains (Saint-Venant's equations), for the plane and the space case, as well as for the membrane with constant curvature. This analogy allows a very simple passage from the integrals to the compatibility equations. For example, in the space case, the general integrals for the equations of equilibrium (author's notations) $F^i \equiv p^{ik}|k=0$ (i, $k=1, 2, 3$) are

$$(1) \qquad p^{ik}=\varepsilon^{irh}\varepsilon^{ksj}\chi_{hj|rs},$$

where p^{ik} is the stress tensor, ε^{irh} the Ricci's tensor, and χ_{hj} is a symmetric tensor which is, a priori, an arbitrary function of the position, and which the author calls the equilibrium tensor. The compatibility equations for the strain tensor in the same case are

$$(2) \qquad \Omega^{ik}=\varepsilon^{irh}\varepsilon^{ksj}\xi_{hj|rs}=0,$$

where ξ_{hj} denotes the strain tensor. Ω^{ik} is called the compatibility tensor.

It is obvious that the passage from (1) to (2) can be effected simply by substituting χ_{hj} for ξ_{hj}. The author derives several conclusions from this fact, and gives an energetic interpretation of the equilibrium and the compatibility tensor. *T. P. Anđelić* (Belgrade).

Finzi, Leo. Legame fra equilibrio e congruenza e suo significato fisico. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 338–342.

In this paper, the author continues to make use of the results quoted in the preceding review. He gives certain generalizations of the results obtained there, and also gives a geometric interpretation of the equilibrium and the compatibility tensor. *T. P. Anđelić.*

Washizu, K. A note on the conditions of compatibility. J. Math. Phys. **36** (1958), 306–312.

Castigliano's theorem is a variational principle for determining that particular equilibrated set of stresses which is compatible with the stress-strain relations of linear elasticity theory. The main result delivered by the principle is that the strains satisfy St. Venant's conditions of compatibility. Southwell's paradox consists of the fact that, if we express the equilibrium of the stresses, not by the differential equations, but by one of the known forms of general solution of those equations, Castigliano's theorem then delivers some, but not all, of the conditions of compatibility.

The author resolves this paradox neatly by observing that St. Venant's conditions are not fully independent. They result from linearization of the condition $R_{ijkl}=0$ for a flat space; since R_{ijkl} satisfies Bianchi's identities, a counterpart of these results, also in the linearized case. As the author notes, these simple identities are easily verified as exact properties of the linear tensor, the vanishing of which yields St. Venant's equations. The author shows that in virtue of these counterparts of the Bianchi identities, St. Venant's equations can be divided into two sets of three, such that the vanishing of one set in the interior and of both sets on the boundary implies the vanishing of the other set in the interior. When the arguments leading to Southwell's paradox are examined, it is found that they, in effect, transfer one of these sets of equations to the boundary; by the author's theorem, it follows that both of the sets are in fact satisfied.

C. Truesdell (Bloomington, Ind.).

Paria, Gunadhar. Axisymmetric consolidation for a porous elastic material containing a fluid. J. Math. Phys. **36** (1958), 338–346.

The author analyzes the stress in a semi-infinite medium containing a compressible fluid in the case where a uniform normal stress is suddenly applied to a circular area. A solution is obtained for a concentrated force by a limiting process. This follows a general discussion of axisymmetric solutions. These are shown to be determined by two functions which must satisfy four different equations, three of which appear to be independent.

J. L. Ericksen (Baltimore, Md.).

Zizicas, G. A. Reduction of three-dimensional stress distributions to two-dimensional analysis by superposition. J. Appl. Mech. **24** (1957), 478–480.

By a method of superposition, the author presents another analytical verification of his proposed graphical procedure for specifying the direction of the total shear on an element of surface [same J. **22** (1955), 273–275]. This method simplifies the analysis considerably.

L. E. Payne (College Park, Md.).

Mişicu, M. Über die Anwendung analytischer Funktionen auf dreidimensionale Probleme der Mechanik verformbarer Körper. Rev. Méc. Appl. **1** (1956), no. 2, 93–113.

The author examines a number of properties of a class of functions in which the variable is expressed in the form of a quaternion. The process is analogous to that used in studying functions of a complex variable. The author states his intention of applying these results to problems in the mechanics of continuous media. *L. E. Payne.*

Chattarji, P. P. A note on torsion of circular shafts of variable diameter. J. Appl. Mech. **24** (1957), 477–478.

A circular shaft of variable diameter may be formed by the revolution of a given curve about an axis. By solving the equation for the torsion function in cylindrical polar coordinates, various curves are found, and the torsion function for shafts having such curves as generating curves is determined. *R. M. Morris* (Cardiff).

Levy, J. C. Deflection of a beam referred to any set of rectangular centroidal axes. J. Appl. Mech. **23** (1956), 464–467.

Higuchi, Seiichi; Saito, Hideo; and Hashimoto, Chiaki. A study of the approximate theory of an elastic thick beam. Canad. J. Phys. **35** (1957), 757–765.

Jones, E. E. Finite Fourier transform analysis of the flexure of a non-uniform beam. J. Roy. Aero. Soc. **60** (1956), 805–806.

Ionov, V. N. Equilibrium of an elastic thick-walled tube under internal pressure. Vestnik Moskov. Univ. **11** (1956), no. 5, 13–24. (Russian)

Wittrick, W. H. Edge stresses in thin shells of revolution. Austral. J. Appl. Sci. 8 (1957), 235–260.

Bekanntlich kann man den Spannungszustand in dünnen Umdrehungsschalen für eine stetige Verteilung der Flächenbelastung und einen stetigen Verlauf des Meridians in guter Annäherung durch eine Überlagerung des Membranspannungszustandes mit einem nur auf die Umgebung der Ränder beschränkten Biegezustand, den man als Randstörung bezeichnet, darstellen. Der Verfasser weist darauf hin, daß die Ermittlung der Membranspannungen prinzipiell immer lösbar ist, notfalls mit Hilfe einer näherungsweisen Integration der Differentialgleichungen nach der Differenzenrechnung. Wenn man dieses Verfahren von vornherein gleich auf die vollständigen Gleichungen der Schalenbiegetheorie anwenden wollte, so müßte man in der Nähe der Ränder wegen der dort stark veränderlichen Spannungen ein sehr engmaschiges Netz wählen, wodurch der Rechenaufwand um ein Vielfaches größer würde. Somit behält die Aufspaltung in Membranlösung und Randstörung in jedem Falle seine Berechtigung, auch wenn die erstgenannte Lösung keine geschlossene Darstellung zuläßt.

Es werden dann geschlossene Formeln zur näherungsweisen Berechnung der Randstörung entwickelt, wobei die Größenordnung der einzelnen Glieder und die des entstehenden Fehlers sorgfältig abgeschätzt werden. Im Gegensatz zu früheren Arbeiten auf diesem Gebiet beschränkt sich der Verfasser nicht nur auf solche Ränder, die von einem Breitenkreis gebildet werden, sondern faßt auch Randangriffe längs eines Meridians ins Auge.

Im ersteren Falle gibt er die bislang bekannte Lösung an, bei welcher der relative Fehler in der Größenordnung von $\sqrt{(t/R_2)}$ liegt, wobei t die Schalenstärke und R_2 den zweiten Hauptkrümmungsradius der Umdrehungsschalen am Rande bezeichnet. Neu ist die erstmalig von Wittrick angegebene sog. 2. Näherung, bei der durch Zusatzglieder zu den Schnittgrößen und Verschiebungen aus der 1. Näherung der Fehler auf die Größenordnung t/R_2 reduziert wird und somit in derselben Größenordnung liegt, wie die an sich in der üblichen linearen Schalentheorie schon enthaltenen Ungenauigkeiten.

Im Falle der von einem Meridian gebildeten Ränder erübrigt sich der 2. Rechnungsgang, da hier, bedingt durch die besondere Geometrie der Umdrehungsschalen, schon der 1. Schritt eine Genauigkeit bis auf Glieder der Größenordnung r_1t/r_0^2 liefert. Hierin bedeutet r_1 den Meridiankrümmungsradius und r_0 den Radius des Breitenkreises. Man ersieht, daß diese Rechnung nur solange brauchbare Ergebnisse liefert, wie r_1/r_2 ungefähr von der Größenordnung 1 ist. Insbesondere versagt dieses Verfahren bei Kegel- und Zylinderschalen, wo $r_1=\infty$ ist.

Abschließend erläutert der Verfasser sein Verfahren an dem Beispiel einer Apsidenschale. *W. Zerna.*

Clark, R. A.; and Reissner, E. On stresses and deformations of ellipsoidal shells subject to internal pressure. J. Mech. Phys. Solids 6 (1957), 63–70.

Die Verfasser untersuchen eine elliptische Umdrehungsschale unter innerem Überdruck, einmal nach der Membrantheorie und vergleichsweise nach der Schalenbiegetheorie, welche der von Reissner [Reissner Anniversary Volume, Edwards, Ann Arbor, Michigan, pp. 231–247; MR 11, 69] für Umdrehungsschalen entwickelten Form zugrunde gelegt wird.

Obwohl die Membrantheorie in der Lage ist, alle an die Lösung zu stellenden Bedingungen zu erfüllen, stimmt sie nur dann mit den Ergebnissen der Biegetheorie weitgehend überein, wenn $h/R_{\min}=ah/b^2\ll b^2/a$ gilt. Hierin bezeichnet h die Schalenstärke, $R_{\min}$ den kleinsten Meridiankrümmungsradius und a bzw. b die große bezw. kleine Halbachse der Meridianellipse.

Falls $b/a\ll 1$ ist, also für sehr flache Schalen, bedeutet diese Bedingung eine wesentlich stärkere Einschränkung als die normalerweise in der Schalentheorie zugrunde gelegte Annahme $h/R_{\min}\ll 1$. In einem Diagramm zeigen die Verfasser für verschiedene Verhältnisse $h/R_{\min}$ und a/b, wie die aus der Membrantheorie und der Biegetheorie für einen charakteristischen Punkt errechneten Spannungen sich zueinander verhalten. *W. Zerna.*

Olesiak, Zbigniew. Discontinuous boundary conditions and linear supports in statical problems of cylindrical shells. Arch. Mech. Stos. 9 (1957), 549–563. (Polish and Russian summaries)

Bekanntlich läßt sich die mathematische Behandlung der Zylinderschalen, wenn man die heute allgemein übliche Theorie für dünne Schalen zugrunde legt, auf die Lösung einer partiellen Differentialgleichung 8. Ordnung für eine Spannungsfunktion zurückführen. Aus dieser Spannungsfunktion folgt dann die Normalverschiebung der Schalenmittelfläche und aus den partiellen Ableitungen beider Größen alle Schnittkräfte.

Lösungen der oben erwähnten Differentialgleichung sind bislang nur für den Fall der allseitig unverschieblich, aber frei drehbar gelagerten Schale bekannt. Der Verfasser zeigt einen Weg, der — von diesen bekannten Lösungen ausgehend — auch eine Behandlung der auf der ganzen Länge der Ränder oder auch nur abschnittsweise starr eingespannten Schale ermöglicht. Schließlich bezieht er auch noch den Fall einer Linienstützung der Schale im Inneren in seine Betrachtungen mit ein.

Hierzu wird die frei gelagerte Schale als Grundsystem benutzt und die entsprechenden Randverdrehungen bzw. Durchbiegungen im Inneren der Schale durch entsprechende Randmomente bzw. Linienlasten zu Null gemacht. Es werden so Fredholmsche Integralgleichungen 1. Art erhalten, deren Kerne aus bekannten Biegeflächen bzw. deren Ableitungen für die frei drehbar gelagerte Schale bestehen.

In einem besonderen Abschnitt werden die wichtigsten Kerne dieser Art durch trigonometrische Reihen explizit dargestellt. Abschließend erläutert der Verfasser die Möglichkeit, die Integralgleichungssysteme durch lineare nichthomogene algebraische Gleichungssysteme zu approximieren. Dieser etwas mühsame Weg scheint die einzige Möglichkeit zu sein, die Formeln numerisch auszuwerten. *W. Zerna* (Hannover).

Seide, Paul. A Donnell type theory for asymmetrical bending and buckling of thin conical shells. J. Appl. Mech. 24 (1957), 547–552.

In view of the success achieved in solving simplified equations for circular cylindrical shells, it appears reasonable to look for comparable equations for other shell geometries. The purpose of this paper was to derive a consistent set of equations for the asymmetrical bending and buckling of conical shells, which are relatively simple yet retain the essential ingredients of limiting cases; e.g., cylindrical shell, flat plate. The approximations are made in the strain displacement relations and the equations and boundary conditions are obtained by variational methods from the total energy. The author's equations, which satisfy the limiting conditions, are expressible as two fourth order equations in the radial displacement w

and the membrane stress function, or as a single complex fourth order differential equation. The buckling equation is also expressible as a single eighth order equation in w and two fourth order equations relating the other displacements to w. Further remarks on transformations of the buckling equations appear in a later note by the author [J. Aero. Sci. **25** (1958), 342]. It is noted by the author that his buckling equations have been previously derived by Russian authors. The author does not attempt to solve the equations in this paper, and it is known that even approximate solutions to such equations are very difficult to obtain. *S. R. Bodner* (Providence, R.I.).

Seth, B. R. Finite bending of a plate into a spherical shell. Z. Angew. Math. Mech. **37** (1957), 393–398. (German, French and Russian summaries)

Using the theory of finite deformations of a compressible elastic material, the author solves the problem of a circular plate bent into a spherical shell segment by the action of edge couples and forces.

The solution obtained is an approximate one, as the only exact solution to the problem corresponds to membrane-type deformation. Expressions are obtained for the edge couple and stress resultant necessary to maintain the spherical shape, and for the change in the plate thickness. The results are illustrated by some numerical examples. *S. R. Bodner* (Providence, R.I.).

Singh, R. K. P. The Green's function of an elliptic plate. Mathematika **4** (1957), 61–69.

The author discusses the Green's function for the biharmonic operator in an elliptic plate on whose boundary the transverse displacement and its normal derivative must vanish. The displacements produced by equal forces at conjugate points $(x_0, \pm y_0)$ are first calculated, and then those due to equal and opposite forces at the same pair of points. The appropriate Green's function is then one half the sum of these two functions; the author states that his representation converges rapidly as long as the ellipse is not too elongated. The centrally loaded elliptic plate is considered in more detail, and comparisons are made with the limiting cases involving a circular plate or an infinite strip. *W. E. Boyce* (Troy, N.Y.).

Pan, Lih-chow. Equilibrium, buckling and vibration of a 30°-60°-90°-triangular plate simply supported at the edges. Sci. Sinica **6** (1957), 347–379.

The torsion function and rigidity of a right-angled prism containing an angle of 60° are obtained by an image method. A similar procedure is adopted for the bending of a simply supported 30°-60°-90° triangular plate subjected to (i) uniform load; (ii) concentrated load. Its buckling and vibrations are discussed, and the results found agree with those obtained by B. R. Seth [Phil. Mag. (7) **38** (1947), 292–297; Bull. Calcutta Math. Soc. **40** (1948), 36–40; MR **9**, 220; **10**, 172] and G. E. Hay [Proc. London Math. Soc. **45** (1939), 382–397; MR **1**, 21].

The general results already known for bending of simply supported plates, of which those obtained by the author are particular cases, are not noted [Seth, Z. Angew. Math. Mech. **35** (1955), 96–99; MR **16**, 1071]. For example, for such a plate, under the combined action of a compressive thrust and a transverse load, which may be concentrated, uniformly distributed or a given function of the coordinates in the plane of the plate, the critical load is the same as the least eigenvalue for the transverse vibrations of a clamped membrane having the same boundary as that of the plate. *B. R. Seth.*

Atsumi, A. Stress concentrations in a strip under tension and containing two pairs of semicircular notches placed on the edges symmetrically. J. Appl. Mech. **24** (1957), 565–573.

Distributions of stress in an isotropic plate of infinite length and finite breadth under tension along its length and containing two pairs of equal semi-circular notches placed symmetrically on the edges are studied theoretically. A method of successive approximation has to be employed to find the coefficients needed in the stress function, and in the numerical examples given the results are shown to be obtainable to any degree of approximation. The results for the stress along the straight and curved boundaries are compared with those for the strip with a single pair of equal semi-circular notches placed symmetrically. *R. M. Morris* (Cardiff).

Horvay, G.; and Hanson, K. L. The sector problem. J. Appl. Mech. **24** (1957), 574–581.

Using the variational method, approximate solutions

$$f_k(r)h_k(\theta),\ f_k(r)g_k(\theta),\ F_k(\theta)H_k(r),\ F_k(\theta)G_k(r)$$

of the biharmonic equation are established for the circular sector with the following properties: The stress functions f_kh_k create shear tractions on the radial boundaries; the stress functions f_kg_k create normal tractions on the radial boundaries; the stress functions F_kH_k create both shear and normal tractions on the circular boundary, and the stress functions F_kG_k create normal tractions on the circular boundary. The factors $f_k(r)$ constitute a complete set of orthonormal polynomials in r into which self-equilibrating normal or shear tractions applied to the radial boundaries of the sector may be expanded; the factors $F_k(\theta)$ constitute a complete set of orthonormal polynomials in θ into which shear tractions applied to the circular boundary of the sector may be expanded; and the functions $F_k''+F_k$ constitute a complete set of non-orthogonal polynomials into which normal tractions applied to the circular boundary of the sector may be expanded. Function tables are also presented. [From author's summary.] *E. H. Mansfield* (Farnborough).

Mihailescu, M. General criteria for the calculation of shells. Rev. Méc. Appl. **1** (1956), no. 2, 139–151.

The question of the appropriateness of the membrane theory of shells for general shells under arbitrary loading is discussed in this paper. An approximate equilibrium equation for shells is derived which compares with those of other authors. The author investigates the particular solution of this equation which, if the loads do not vary greatly, would be the solution for the membrane theory. By showing the similarity between the equation of characteristics of the membrane equilibrium equation and the equation governing the shell geometry, the author correlates the classes of shells with the solutions obtainable from the membrane theory. The importance of the non-satisfaction of the exact membrane boundary conditions on the solution is then discussed for the various classes of shell geometries. *S. R. Bodner.*

★ **Vlasov, V. Z. General theory of shells. Chap. X. More complex problems of shallow shell theory, pp. 436–478.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 51 pp.

A translation of Chapter X of the book listed in MR **11**, 627.

★ **Vlasov, V. Z. General theory of shells. Chap. IX. Shallow spherical shells, pp. 398–436.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 48 pp.

A translation of Chap. IX of the book listed in MR 11, 627.

★ **Tsurkov, I. S. On elasto-plastic equilibrium of shells of revolution for small axisymmetric deformations.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 7 pp.

Translated from Izv. Akad. Nauk SSSR Otd. Tehn. Nauk **1956**, No. 11, 106–110.

Kappus, Robert. Strenge Lösung für den durch zwei Einzelkräfte belasteten Kreisring. Z. Angew. Math. Mech. **35**, 210–231 (1955). (English, French and Russian summaries)

Finkel'šteĭn, R. M. On a problem of statics for a thin cylindrical shell. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 5, 136–140. (Russian)

Hemp, W. S. Notes on the problem of the optimum design of structures. Coll. Aero. Cranfield. Note no. 73 (1958), i+8 pp.

Fleĭšman, N. P. Bending of an infinite plate with a reinforced circular opening. L'vov. Gos. Univ. Uč. Zap. **29**, Ser. Meh.-Mat. no. 6 (1954), 105–111. (Russian)

Mader, F. W. Beitrag zur Berechnung in Querrichtung durchlaufender Plattenstreifen mit Hilfe Fourierscher Integrale. Ing.-Arch. **25** (1957), 201–204.

Müggenburg, H. Einflussflächen für die am bogenförmigen Rand eingespannte und am geraden Rand freie Halbkreisplatte. Ing.-Arch. **24** (1956), 308–316.

★ **Fung, Y. C. Flutter of curved plates with edge compression in a supersonic flow.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 221–245. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

To explain a significant quantitative discrepancy between previous theory and experiments on the large-amplitude flutter of buckled plates in a supersonic flow, the effects of the initial curvature and of a static pressure differential across the plate are calculated. It is found that the order of magnitude of the initial deviation from flatness required to account for the discrepancy is quite reasonable, so that the explanation may very well lie in that direction. *From the author's summary.*

★ **Targoff, Walter P. The bending vibrations of a twisted rotating beam.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 177–194. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

In order to understand more fully the results of an experimental program investigating the dynamic behavior of propeller blades rotating in vacuo, an analytic method has been developed which permits the computation of the natural modes and frequencies of twisted rotating beams. The method employs mathematics and symbology easily comprehended by the average vibration engineer. The calculations are well suited for programming on a computing machine of the IBM C. P. C. type. *From the author's summary.*

Filippov, A. P. Oscillations of a beam under the influence of a moving load. Akad. Nauk Ukrain. RSR. Prikl. Meh. 1 (1955), 268–275. (Ukrainian. Russian summary)

★ **Lo, Hsu; and Danforth, C. E. On rotating blades with flexible mounting.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 160–176. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

Ziller, Felix. Über die Flatterschwingungen von Hängebrücken. VDI Z. 99 (1957), 405–415.

Hoff, N. J. Buckling of thin cylindrical shell under hoop stresses varying in axial direction. J. Appl. Mech. **24** (1957), 405–412.

Der Verfasser untersucht die Stabilität dünner, geschlossener kreiszylindrischer Schalen, die an ihren Enden so gestützt sind, daß dort keine Verschiebungen in Radial- und Umfangrichtung, wohl aber in axialer Richtung stattfinden können. Desgleichen soll eine unbehinderte Drehung um die Randtangente möglich sein. Die Beanspruchung der Schalen erfolgt so, daß eine in der Längsrichtung veränderliche Ringdruckspannung entsteht. Dieser Zustand kann beispielsweise durch eine in der Längsrichtung ungleichförmige Erwärmung hervorgerufen werden; andererseits kann er auch durch eine gleichmäßige Erwärmung bzw. einen gleichförmigen äußeren Druck erzeugt werden, wenn die Schale in gewissen Abständen durch Rippen, die die Verformung behindern, ausgesteift wird.

Ähnliche Probleme sind früher schon durch von Sanden und Tölke auf der Grundlage der Loveschen Formeln behandelt worden. Im Gegensatz hierzu entwickelt der Verfasser seine Theorie mit Hilfe der sog. Donnellschen Gleichungen, wie sie heute auch fast ausschließlich der normalen Gleichgewichtsuntersuchung der Zylinderschalen zugrunde gelegt werden. Lediglich die Differentialgleichungen 8. Ordnung für die Normalverschiebung w enthält noch ein mit der Ringspannung σ_φ des Grundzustandes multipliziertes Glied, welches daher rührt, daß man bei der Untersuchung der Stabilität bekanntlich die Gleichgewichtsbedingungen am verformten System anschreiben muß, so daß Produkte der Schnittgrößen und Verschiebungen auftreten. In üblicher Weise werden die Verschiebungen als Fouriersche Doppelreihen so angesetzt, daß die Randbedingungen an den Enden automatisch erfüllt sind. Auch σ_φ wird als Fourier-Reihe nach der Längskoordinate x entwickelt; in der Umfangrichtung wird σ_φ konstant angenommen. Zunächst ist es möglich, sämtliche Koeffizienten in den Reihen für die Verschiebungen durch die Koeffizienten der Reihe für die Normalverschiebung auszudrücken, die somit als einzige Unbekannte verblieben. Letztere bestimmen sich aus der Forderung, daß die erwähnte Differentialgleichung 8. Ordnung für w identisch erfüllt sein muß. Das führt im allgemeinen Fall zu einem linearen, homogenen System von unendlich vielen Gleichungen für unendlich viele Unbekannte, welches nur dann nichttriviale Lösungen haben kann, wenn die zugehörige Determinante verschwindet.

In allgemeiner Form läßt sich das Problem natürlich

nicht lösen. Man kann sich aber in der Reihenentwicklung für σ_φ auf wenige Glieder beschränken. Der Sonderfall einer auch in der Längsrichtung gleichförmigen Verteilung läßt sich dann relativ leicht behandeln. Aber schon, wenn man σ_φ in Form einer Halbwelle ansetzt, wird die Lösung schwieriger und kann nur noch approximativ erfolgen.

Der Verfasser weist sodann nach, daß die gewonnenen Näherungswerte alle über dem exakten Beulwert liegen, aber rasch gegen diesen Grenzwert konvergieren.

Abschließend wird noch das Ausbeulen unter einer gleichförmigen Temperaturerhöhung behandelt; durch die Einschnürung an den Enden entstehen auch hier Druckspannungen, die zur Mitte hin allerdings rasch abfallen, aber in den Randbereichen erhebliche Werte annehmen, so daß ein Ausbeulen theoretisch möglich ist. Allerdings ist die kritische Temperatur, die das Ausbeulen bewirkt, so hoch, daß sicher schon vorher plastische Verformungen stattfinden, die das Ausbeulen verhindern, wie der Verfasser richtig bemerkt. *W. Zerna* (Hannover).

Bodner, Sol R. On the conservativeness of various distributed force systems. J. Aero. Sci. **25** (1958), 132–133.

Buckling of a circular ring under three similar types of conservative force fields is considered. The cases considered are hydrostatic pressure, constant directional pressure, and centrally directed pressure. Although the differences in the force field are slight, the corresponding buckling loads are different. The order of the cases considered is the same as the order of magnitude of the buckling loads. *G. H. Handelman* (Troy, N.Y.).

★**Ivlev, D. D. Buckling of eccentric pipes.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 7 pp.

Translated from Izv. Akad. Nauk SSSR Otd. Tehn. Nauk **1956** No. 10, 112–116.

Rabotnov, Yu. N.; and Šesterikov, S. A. Stability of rods and plates under conditions of creep. Prikl. Mat. Meh. **21** (1957), 406–412. (Russian)

Sternberg, E.; and Eubanks, R. A. On stress functions for elastokinetics and the integration of the repeated wave equation. Quart. Appl. Math. **15** (1957), 149–153.

A completeness proof is given for the generalized Galerkin solution of the equations of classical elastokinetics, and the solution is related to a generalization of Papkovich's solution of the equilibrium equations. When body forces are zero, the Galerkin vector of elastokinetics satisfies a repeated wave equation. The authors prove that every solution of such an equation may be represented as the sum of two wave functions. *A. E. Green.*

Noll, W. Verschiebungsfunktionen für elastische Schwingungsprobleme. Z. Angew. Math. Mech. **37** (1957), 81–87. (English, French and Russian summaries)

The problem solved here is included in a more general one solved in the paper reviewed above, which was not available to the author.
C. Truesdell (Bloomington, Ind.).

Boillet, Pierre. Sur l'interprétation du principe de Huygens: cas des ondes acoustiques, élastiques et électromagnétiques. Cahiers de Phys. **11** (1957), 238–268.

This is chapter II of a paper in three parts on Huygens' Principle. Chapter I [Cahiers de Phys. no. **78** (1957), 59–87; MR **19**, 497] dealt with acoustic waves; Chapter III [MR **19**, 1011] will deal with electromagnetic waves. The present chapter is concerned with elastic waves.

It is not possible to give a brief account of the work. In effect, the author is not content with an analytical formulation of Huygen's Principle; it must also have an evident physical interpretation. His procedure is roughly the following.

Suppose we are given the sources of an elastic disturbance inside some closed surface σ_i. Let σ_e be another closed surface enclosing σ_i. Then it is possible to construct a distribution of sources in the volume bounded by σ_i and σ_e which will have no effect inside σ_i and will have the same effect outside σ_e as the primitive sources inside σ_i; and, morover, the effect of this volume distribution of sources can be expressed as a function of the disturbance on σ_e. This gives an analytical formulation of Huygens' Principle. But to get a physical interpretation, it is necessary to consider the effect of making the thickness of the shell bounded by σ_i and σ_e tend to zero. This limiting process leads to the introduction of "pseudo-sources" on σ_e; these are of three types. A source of type (i) represents a creation of matter, and is necessary because matter crosses each fixed surface element in an elastic vibration. A source of type (ii), called a torsion source, represents the state of torsion in which the matter crossing a fixed surface element is. A source of type (iii) represents the mechanical interaction of the two parts of the solid separated by a given surface element. *E. T. Copson.*

Mirsky, I.; and Herrmann, G. Nonaxially symmetric motions of cylindrical shells. J. Acoust. Soc. Amer. **29** (1957), 1116–1123.

In an earlier paper [J. Appl. Mech. **23** (1956), 563–568; MR **19**, 905], the authors studied the propagation of axisymmetric free harmonic waves in the axial direction of a shell of uniform thickness and infinite extent. Their approach differs from the classical small-deflection theory for thin shells by the inclusion of the effects of transverse shear deformation and rotatory inertia, in the manner originally introduced by Mindlin for flat plates.

The present paper extends the earlier work to include non-axially-symmetric motions of the shell. After deducing the general equations of motions for free dynamical motions, the specific case of waves harmonic in the axial direction (and in time) is studied. Numerical evaluations of the characteristic equation were accomplished on a magnetic drum calculator for the following ranges: Ratio of shell thickness to mean radius of 1/30, 1/10 and 1/4 (the latter merely to indicate thick-shell trends); number of circumferential waves from 0 to 6.

The results given are of considerable interest and value. Although experimental data are not included, it is reasonable to suppose that the present theory will be applicable over the range from quite moderate wave-lengths to the classical range of long wave-lengths.
M. Goland (San Antonio, Tex.).

Cooper, R. M.; and Naghdi, P. M. Propagation of nonaxially symmetric waves in elastic cylindrical shells. J. Acoust. Soc. Amer. **29** (1957), 1365–1373.

Two general systems of displacement equations of motion, designated as (I) and (II), have been derived previously for thin, circular, elastic cylindrical shells by the authors [J. Acoust. Soc. Amer. **28** (1956), 56–63]; the equations include the effects of transverse shear de-

formation and rotatory inertia and have been used to study torsionless, axisymmetric wave propagation in an infinite cylindrical shell. (On neglecting the above effects, equations (I) reduce to those known as Love's first approximation, while equations (II) correspond to those of Donnell.)

This paper predicts phase velocities and amplitude ratios for nonaxially symmetric, as well as purely torsional, propagation of elastic waves in an infinite cylindrical shell; the results obtained by using the two systems of equations are compared and necessary restrictions on the use of equation (II) suggested. *G. B. Warburton.*

Kristesku, N. Propagation of waves along flexible chains (influence of the speed of deformation). Prikl. Mat. Meh. **21** (1957), 486–490. (Russian)

The velocity of wave propagation along a flexible chain of an elastic material is investigated with the aid of the characteristics of the differential equation.

A. M. Freudenthal (New York, N.Y.).

Chopra, S. D. The range of existence of Stoneley waves in an internal stratum. II. Antisymmetric vibrations. Monthly Not. Roy. Astr. Soc. Geophys. Suppl. **7** (1957), 338–346.

This is a continuation of paper reviewed in MR 19, 104; the theory has been extended to cover antisymmetric vibration.

Jeffreys, Harold. Elastic waves in a continously stratified medium. Monthly Not. Roy. Astr. Soc. Geophys. Suppl. **7** (1957), 332–337.

★ **Plass, H. J., Jr. Damping of vibrations in elastic rods and sandwich structures by incorporation of additional viscoelastic material.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 48–71. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

★ **Hilton, Harry H. Pitching instability of rigid lifting surfaces on viscoelastic supports in subsonic or supersonic potential flow.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 1–19. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

Finzi, Leo. On the principle of Haar and von Karman in statically determinate problems of plasticity. J. Appl. Mech. **24** (1957), 461–463.

The Haar-von Kármán principle, proved by Greenberg, is related to a partly plastic solid defined by the von Mises yield condition and the Hencky relations between stress and total strain. It states that, if the surface tractions are held fixed, and the internal stress field is varied subject to the equations of equilibrium and the yield inequality, the elastic complementary energy of the whole body has an absolute (though not analytic) minimum in the actual state, and is stationary for infinitesimal variations of stress giving the same plastic zone. The present paper is concerned only with infinitesimal variations in the very special situation where the stress in the plastic zone is completely determined by the surface tractions and yield condition alone; only the elastic field can then be varied. From the illustrative worked example (thick tube under pressure), it seems that the author wishes to prove an extended stationary principle for the case when the plastic boundary is also varied, with the yield condition possibly unsatisfied in the infinitesimal strip between the original and final positions. The reviewer considers the proof given to be inadequate, though in fact, the result follows by an easy extension of Greenberg's own proof. It is also stated that the extension is valid regardless of the material properties; the reviewer considers that the yield surface must be convex, and the plastic potential law must hold.

R. Hill (Nottingham).

Życzkowski, Michał. Limit state of non-homogeneous rotating discs. Rozprawy Inż. **5** (1957), 49–96. (Polish. Russian and English summaries)

The plane stress problem of the plastic rotating circular disk with variable thickness and/or inhomogeneous yield condition $f(\sigma_r, \sigma_\theta)=k(r)$ is investigated under the assumption of the von Mises and of the Tresca formulation of the yield condition. Obviously, the latter permits direct integration, while the former requires approximate methods. One of those dealt with is an inversion method in which an arbitrary statically admissible state of stress is assumed, and the associated function $k(r)$ determined. The deformations are determined on the basis of Hencky's stress-strain relations. *A. M. Freudenthal.*

Ivlev, D. D. Approximate solution of elastic-plastic problems of the theory of ideal plasticity. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 294–296. (Russian)

Attention is given to the approximate determination of the stress distribution in the vicinity of a stressed boundary under conditions of ideal plastic plane strain. The method assumes the representation of stresses as power series in a parameter. Some examples are considered.

H. G. Hopkins (Sevenoaks).

Drucker, D. C. The effect of shear on the plastic bending of beams. J. Appl. Mech. **23** (1956), 509–514.

★ **Gatewood, B. E. Thermal stresses.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1957. xi+232 pp. $7.50.

This book is written primarily for the practising aeronautical structural engineer and it provides a valuable introduction to the problem of estimating thermal stresses and their effects. The emphasis throughout is on the determination of realistic values, coupled with the use of simplifying assumptions. The subject matter includes: the general thermal stress-strain equations in three dimensions; the stress function for two-dimensional thermal stresses and methods of solution employing variational methods and complex variables; the equations of heat conduction, radiation and convection with notes on aerodynamic heating; a simplified analysis of heated joints and skin-stringer combinations; one-dimensional solutions of transient heating; thermal buckling; allowable stresses at elevated temperatures. There are over 300 references.

E. H. Mansfield (Farnborough).

Klosner, J. M.; and Forray, M. J. Buckling of simply supported plates under arbitrary symmetrical temperature distributions. J. Aero. Sci. **25** (1958), 181–184.

The stress distribution is found for a rectangular plate, whose supported edges remain straight, subjected to an arbitrary symmetrical temperature distribution. The Rayleigh-Ritz procedure is then used to determine approximately when buckling occurs.

E. H. Mansfield (Farnborough).

Bader, W. Zur numerischen Bestimmung der Wärmespannungen. Z. Angew. Math. Mech. 36 (1956), 331–339. (English, French and Russian summaries)

A steady-state thermoelastic displacement field has been split into two parts: a pure irrotational, and a mixed one. For the first field, some particular integrals of the Poisson equation, concerning the thermoelastic potential ψ, are given in the form $\psi=T\cdot h(s)$, where T is the temperature and $h(s)$ the solution of an ordinary differential equation of the second order (the variable s depending on the type of the coordinate system). For the displacements of the second field, which serves to meet the boundary conditions, a representation in the form of some series with unknown coefficients is proposed. A criterion of minimum deviation of the surface tractions from their prescribed values is given. A numerical example is solved, concerning a circular finite cylinder with the bases maintained at constant temperatures. *J. Nowinski.*

Thrun, Zygmunt. Thermal deformations and stresses in thin rectangular and circular plates of variable thickness. Rozprawy Inż. 4 (1956), 523–541. (Polish. Russian and English summaries)

The differential equation of the deflection of a thin plate of variable thickness with linear temperature variation between the top and bottom surfaces is solved for a rectangular plate with linearly variable thickness, freely supported or fixed along two parallel edges, as well as for a circular plate with axially symmetric parabolic variation of thickness, by the standard expansion of the deflections into trigonometric series. A formal solution for the circular plate is also obtained by the use of finite differences, with no consideration given to the accuracy of the approximation. *A. M. Freudenthal.*

Tremmel, E. Über die Anwendung der Plattentheorie zur Bestimmung von Wärmespannungsfeldern. Österreich. Ing.-Arch. 11 (1957), 165–172.

The state of stress in a singly-connected prismatical body with free surfaces subjected to a two-dimensional temperature field is investigated. The Airey stress function satisfying the boundary conditions is found to be equivalent to the bending surface of a congruent, clamped plate whose loading is proportional to the Laplacian of the temperature field. *H. D. Conway* (Ithaca, N.Y.).

Boley, B. A.; and Barber, A. D. Dynamic response of beams and plates to rapid heating. J. Appl. Mech. 24 (1957), 413–416.

The authors study thermally induced vibrations of rectangular plates and beams under various types of heat application. They conclude that the role of inertia is important for rapidly applied heat inputs and for thin plates. *L. E. Payne* (College Park, Md.).

Mossakowski, Jerzy. The state of stress and displacement in a thin anisotropic plate due to a concentrated source of heat. Arch. Mech. Stos. 9 (1957), 565–577. (Polish and Russian summaries)

See also: Optics, Electromagnetic Theory, Circuits: Boillet.

Structure of Matter

Goldstein, Louis. On the theory of liquids. Ann. Physics 1 (1957), 33–57.

The theory of liquids is today an important branch of theoretical physics and theoretical chemistry, with an extensive literature which has, with difficulty, been compressed in four or five recent monographs, each dealing with a special aspect. This paper cites references to a few works of the author and three early papers of Zernike and his associates. Some of his results will appear familiar to workers in the field; others will appear strange. The latter involve definite integrals which are usually regarded as divergent. The reviewer has tried without success to determine sufficient conditions for the convergence of such integrals, and is therefore surprised to find the author at the end stressing "the wide generality of the results obtained". *H. S. Green* (Dublin).

Belov, N. V.; Neronova, N. N.; and Smirnova, T. S. Shubnikov groups. Akad. Nauk SSSR. Kristallografiya 2 (1957), 315–325. (Russian)

This paper consists mainly of a tabulation of 1651 Shubnikov groups. A preface states some of the results concerning Shubnikov groups which lead to the derivation of the table and to the classification system. Drawings of three-dimensional two-colored translational lattices are provided for 36 of the 230 Fedorovski groups from which the Shubnikov groups are generated. *E. J. Cogan.*

Fieschi, R. Matter tensors in the crystallographic groups of Cartesian symmetry. Physica 23 (1957), 972–976.

A table of the components of general tensors of polar and axial nature, of second, third and fourth order is given for the 20 Cartesian groups: 1, $\bar{1}$, 2, m, $2/m$, 222, $mm2$, mmm, 4, $\bar{4}$, $4mm$, 422, $\bar{4}2m$, $4/m$, $4/mmm$, 23, $m3$, 432, $\bar{4}3m$, and $m3m$. *W. Nowacki* (Bern).

Elcock, E. W. The cooperative behaviour of a two-dimensional defect crystal. Proc. Cambridge Philos. Soc. 53 (1957), 863–869.

The two-dimensional Ising model of a ferromagnet is modified by assuming that any spin will have the usual Ising type interaction with a nearest neighbor with probability p but no interaction with probability $1-p$. The usual Ising model corresponds to $p=1$. The author is led to the conclusion that the specific heat for this model has a logarithmic singularity which occurs at lower and lower temperatures as p decreases from 1 and disappears at $T=0$ when $p=-1+\sqrt{2}$ is still positive.

These interesting results are probably incorrect, however, because in the derivation, quantities that should be treated as random variables are indiscriminately replaced by their expectations in places where this may lead to serious errors. *G. Newell* (Providence, R.I.).

Deigen, M. F. Theory of localized electron states in an isotropic homopolar crystal. Soviet Physics. JETP 4 (1957), 424–430.

The author introduces the so-called condenson interaction of an electron, localized near an impurity center, with the lattice vibrations of an isotropic, homopolar crystal. He calculates the energy levels of the system and the energy of thermal dissociation of the electron both classically and quantum mechanically. He determines the shape of the absorption band due to a localized electron and works out in detail the case of a Coulomb potential for the defect.

H. A. Hauptman (Washington, D.C.).

Bronskiĭ, A. P. Velocity of deformation of a hollow cylinder under internal pressure. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **11** (1956), no. 1, 13–16. (Russian)

See also: Quantum Mechanics: Lippmann; Selivanenko.

Fluid Mechanics, Acoustics

Howard, Louis N. Divergence formulas involving vorticity. Arch. Rational Mech. Anal. **1** (1957), 113–123.

Most vorticity conservation theorems rest upon an identity of the type (A) $f(x, q, \Omega)=\operatorname{div} V(x, q, \Omega)$, where x is the position vector, q is the velocity field, and $\Omega=\nabla\times q$ is the vorticity. When such an identity is known, use of Green's Theorem combined with appropriate boundary conditions leads to a conservation law. The author sets himself the task of finding all identities of the type (A) and analogous formulae in the plane.

First, he finds that the most general formula of the type (A) in which $f=0$ when $\Omega=0$ is given by the alternative forms

$$\text{(B)} \qquad \nabla A\cdot\Omega=\operatorname{div}(A\Omega)=\operatorname{div}(q\times\nabla A),$$

where $A=A(x)$. Choosing A as the general homogeneous polynomial of degree $n+1$ yields the reviewer's conservation theorem for the nth symmetrical moment of vorticity [Canad. J. Math. **3** (1951), 69–86; MR **14**, 273]. In a formal sense, it is thus shown that the most general vorticity conservation theorem may be obtained by linear combination of the reviewer's results. {Reviewer's note: the simple identity (B) was used by Kelvin; cf. §§ 10, 25, 66 of the reviewer's "Kinematics of vorticity" [Indiana Univ. Press, 1954; MR **17**, 678]. What the author contributes is the proof that all vorticity theorems of the type envisaged are obtained by use of (B).}

When it is demanded that a theorem of type (A) hold in a restricted class of motions, a greater variety of results becomes possible. For isochoric motions, $\operatorname{div} q=0$. Here the author shows that, in addition to identities of the type (B) and to identities due to Lamb and Poincaré, there exist as many more linearly independent formulae of degree n as there are homogeneous harmonic polynomials of degree n, and these formulae he exhibits.

The author obtains corresponding results for the plane. In the isochonic case, infinitely many conservation laws in addition to Hamel's are found. Finally, the author formulates an analogous problem for conservation of the rate of change of vorticity integrals. Here he shows that all such results follow by linear combination of the theorems of the reviewer and of Moreau.

This paper may be said to close much of the problem of general conservation theorems for vorticity.

C. Truesdell (Bloomington, Ind.).

Noll, Walter. On the rotation of an incompressible continuous medium in plane motion. Quart. Appl. Math. **15** (1957), 317–319.

It is shown that if a plane motion is dynamically possible, then so also is the plane motion derived from it by the superposition of a uniform rotation about a fixed axis. This was first proved by G. I. Taylor for perfect liquids, and his results have since been extended to some types of viscous liquid; it is extended in this paper to any incompressible medium such as, for instance, an anisotropic solid.

W. R. Dean (London).

★ **Hirsch, R. Installation de la portance sur un profil muni d'une fente de soufflage au bord de fuite et correction due à l'envergure finie.** Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. **69** (1957), ii+75 pp. 1000 francs.

This is an attempt to extend unsteady-airfoil theory to the case of an airfoil with a "jet flap". The two-dimensional case is considered first. The pressure difference across the jet is related to its momentum and curvature by the result attributed here to Malavard. [No reference given; presumably Rept. NT4/1727A of Office Nationale d'Études et de Recherches Aéronautiques (1954)] Both airfoil and jet-wake are replaced by vortex sheets and the first-order approximation of plane sheets is introduced. The customary relation between wake-vortex intensity at trailing edge and rate of change of circulation is altered by the presence of a term involving the jet curvature there. There is some difficulty in calculating this curvature. A generalization of the Joukowski airfoil is therefore used, such that two critical points appear near the trailing edge instead of the usual one at the trailing edge. This is supposed to permit curvature of the streamline leaving the trailing edge. The result of this calculation is an integral equation analogous to the classical Wagner equation for the circulation about an airfoil started from rest. It is stated that this has been solved numerically, but that the results are clearly in disagreement with experimental results. The author believes this is due to viscous damping; hence, he undertakes to introduce exponential decay of vortex strength. [See Prandtl, Aerodynamic theory, v. III, W. F. Durand, ed., Springer, Berlin, 1935, pp. 34–208]. This requires a new investigation of the motion of free vortices in the wake with damping. Since this calculation involves superposition of the velocity fields of decaying vortices, i.e., application of the single-vortex result to a sheet, and even representation of their fields in complex form, the results surely cannot be accepted without further justification. Numerical calculations indicate rapid decay of vortex speed relative to the airfoil. The Wagner-type integral equation is therefore generalized to account for this effect, and now reasonable agreement with available (steady-flow) results is claimed. It is found that the time (distance) lag of circulation growth is much greater for airfoils with trailing-edge jets than the Wagner result.

Part II is an extension of all this to the wing of finite span. There are now longitudinal vortices in the jet-wake. The author states "Demeurant strictement intérieurs à la nappe du jet formant coupure, leurs effets d'induction ne s'étendent pas au domaine extérieur à celle-ci", which does not make sense to this reviewer. The resulting equations are solved approximately and some numerical results are given. There are a number of appendices giving detailed calculations and numerical values.

This report has been translated and published, apparently completely, in Aircraft Engineering **29** (1957), 366–375; **30** (1958), 11–19.

W. R. Sears (Ithaca, N.Y.).

Leehey, Patrick. The Hilbert problem for an airfoil in unsteady flow. J. Math. Mech. **6** (1957), 427–453.

The linearized problem of incompressible flow about a thin airfoil in arbitrarily accelerated motion is viewed as one of determining the complex perturbation velocity Φ of the fluid as a function holomorphic exterior to a line of discontinuity representing the airfoil and its wake, which is taken as the real axis. This leads to the Hilbert problem of determining the sectionally holomorphic function Φ

which vanishes at infinity and satisfies the condition $\Phi^+=G\Phi^-+g$ on the real axis. The functions G and g have discontinuities at points corresponding to the leading and trailing edges of the airfoil, and to the end point of the wake. For arbitrarily prescribed airfoil and wake discontinuities, Φ is determined uniquely by the condition that it remains bounded in a neighbourhood of the trailing edge. General lift and pitching moment expressions are obtained and applied to special cases of (i) a step change in the angle of attack; (ii) a translatory oscillation; and (iii) an airfoil in an oscillating moving stream.

L. M. Milne-Thomson (Madison, Wis.).

Viswanadha Sarma, L. V. K. Rotational flow of a liquid past a regular polygonal cylinder. Proc. Indian Acad. Sci. Sect. A. **46** (1957), 224–231.

The rotational flow of an inviscid, incompressible fluid past a cylinder of regular polygonal cross-section is investigated with the use of the method of conformal transformation. The fluid thrust on the cylinder is calculated. The results show that the thrust (which is only a lift force in this case) per unit area of cross-section of the cylinder diminishes as the number of sides of the polygon is increased. The corresponding problem for hypotrochoidal cylinders is also considered, and the results indicate that the lift is less for cylinders with slightly curved edges than for those having straight edges.

B. R. Seth (Kharagpur).

Alblas, J. B. On the generation of water waves by a vibrating strip. Appl. Sci. Res. A. **7** (1958), 224–236.

A thin flat plate of infinite length and finite width is partially immersed in an infinitely deep fluid under gravity; its lower edge is at a constant depth below the mean free surface; its upper edge is vertically above the lower edge. The plate is given a simple harmonic flexural vibration of small amplitude about this mean position. The velocity normal to the plate is prescribed and is periodic along the length of the plate. The problem is to find the resulting wave motion. A singular integral equation is obtained for the horizontal velocity below the plate. This can be solved exactly when the wave length along the plate is infinite (two-dimensional motion), as was first done by the reviewer [Quart. J. Mech. Appl. Math. **1** (1948), 246–252; MR **10**, 165] and approximately when the wave length along the plate is small compared to the width of the strip. When the wave length $2\pi/k$ along the strip is connected with the period $2\pi/\omega$ by the relation $\omega^2=gk$, the inviscid theory predicts an infinite wave amplitude and infinite rate of energy transport away from the strip. *F. Ursell* (Cambridge, Mass.).

Proudman, J. On the series that represent tides and surges in an estuary. J. Fluid Mech. **3** (1958), 411–417.

The author considers the propagation of waves of finite amplitude and arbitrary shape, in an infinite channel of uniform cross-section. Frictional effects are neglected. An implicit solution was given by Saint-Venant [C. R. Acad. Sci. Paris **73** (1871), 147–154, 237–240]; in this paper, explicit series solutions for the fluid velocity and surface displacement are developed, using a theorem due to Lagrange. These are specialized to the cases in which there is no permanent current, and the surface elevation at the mouth is prescribed in terms of its harmonic components.

O. M. Phillips (Baltimore, Md.).

Birkhoff, Garrett; and Carter, David. Rising plane bubbles. J. Math. Mech. **6** (1957), 769–779.

The authors treat the plane flow problem of an isolated bubble rising steadily under the influence of gravity in an infinitely long vertical tube of incompressible, inviscid fluid. Adapting methods of Levi-Civita and Villat, they use techniques of conformal mapping to reduce this free boundary problem to a non-linear integral equation. The integral equation is attacked numerically by a procedure based on equal-spaced interpolation and truncation of certain Fourier series. However, in contrast with situations without gravity, the convergence is found to be not really satisfactory. Several unexpected difficulties are brought to light and discussed in the hope that they may be resolved by further analysis. *P. R. Garabedian.*

Miles, John W. On the generation of surfave waves by shear flows. J. Fluid Mech. **3** (1957), 185–204.

This paper considers the stability of an air-water interface under the influence of a wind, neglecting viscosity in the air and the water. It is found that the rate of transfer of energy to a wave of speed c is proportional to the curvature of the velocity profile $U(y)$ in the air at the point where $U=c$. Attention is concentrated on the case where the mean velocity has the logarithmic profile associated with turbulent flow, and an approximate solution for the rate of growth of the waves is found from the inviscid Orr-Somerfeld equation. By considering an energy balance with laminar dissipation in the water, the author predicts that the minimum wind speed for the initiation of gravity waves by this mechanism is of order 100 cm/sec, in agreement with observations by Jeffreys [Proc. Roy. Soc. London. Ser. A. **107** (1925), 189–206]. The mean surface drag predicted is of the order of magnitude observed by Van Dorn [J. Marine Res. **12** (1953), 249–276].

This mechanism and one recently proposed by the reviewer [J. Fluid. Mech. **2** (1957), 417–445; MR **19**, 488] appear to be complementary, but the parts played by each in nature are not yet fully understood. This must await further observation; it is certain, however, that this paper represents a most significant contribution to our understanding of this problem. *O. M. Phillips.*

Kaplun, Saul; and Lagerstrom, P. A. Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers. J. Math. Mech. **6** (1957), 585–593.

The steady motion of incompressible viscous liquid flowing with velocity U past a fixed body is analysed in terms of two sets of non-dimensional variables, Oseen and Stokes variables. Oseen space variables are defined by $\tilde{x}_i=Ux_i/\nu$, ($i=1, 2, 3$), and Stokes variables by $x_i^*=x_i/L$; L is a characteristic length of the body, ν is the kinematic coefficient of viscosity, and the Reynolds number R is UL/ν.

The non-dimensional velocity $\mathbf{q}^*$ is expressed as an Oseen expansion

$$\mathbf{q}^*\sim\sum_{j=0} \varepsilon_j(R)\mathbf{g}_j(\tilde{x}_i),$$

and as a Stokes expansion

$$\mathbf{q}^*\sim\sum \varepsilon_j\mathbf{h}_j(x_i^*),$$

where, for instance, $\mathbf{g}_0=\lim \mathbf{q}^*$ as $R\to 0$ for a fixed nonzero $\tilde{x}_i$. Then two expansions are matched, and the boundary conditions for $\mathbf{g}_j$ and $\mathbf{h}_j$ completed, by a third expansion, the intermediate expansion,

$$\mathbf{q}^*\sim\sum \varepsilon_j\mathbf{u}_j;$$

here $|\mathbf{q}^*-\mathbf{u}_0|\to 0$ as $R\to 0$ in a range of space variables depending on R and overlapping, if possible, the ranges associated with the two earlier limits. This theory is applied to flow past a sphere. In this case $\mathbf{g}_0$ is the unit vector $\mathbf{i}$, while $\mathbf{u}_0$ and $\mathbf{h}_0$ are equal and give the (non-dimensional) Stokes solution. The second approximation is completed by the derivation of $\varepsilon_1, \mathbf{g}_1, \mathbf{h}_1$. *W. R. Dean.*

Kaplun, Saul. Low Reynolds number flow past a circular cylinder. J. Math. Mech. **6** (1957), 595–603.

The theory of the previous paper is here applied to flow past a circular cylinder, and to some general considerations of flow past a cylinder of any cross-section. With the circular cylinder, the asymptotic expansion of $\mathbf{q}^*$ is in terms of powers of ε, where $\varepsilon^{-1}=\log(8/R)+\frac{1}{2}-\gamma$.

W. R. Dean (London).

Lagerstrom, P. A. Note on the preceding two papers. J. Math. Mech. **6** (1957), 605–606.

★ **Gandin, L. S.; and Soloveichik, R. E. On the problems of a laminar boundary layer near a porous wall.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 4 pp.

Translation of Prikl. Mat. Meh. **20** (1956), 663–665.

Spalding, D. B. Transport processes between fluids and clouds of suspended particles: Some exact solutions. Proc. Roy. Soc. London. Ser. A. **242** (1957), 430–443.

The paper is concerned with processes in which solid particles suspended in a fluid have a property, e.g. size, whose rate of change is the product of a function of the property and a function of the local properties of the fluid. The processes proceed uniformly in time and involve convection of fluid and particles from one part of the flow to another and turbulent diffusion, which is represented by an eddy diffusion coefficient dependent on local conditions. The resulting equation for conservation of concentration is then transformed to a form equivalent to the unsteady-state heat-transfer equation, using a particle "age" as the independent variable corresponding to time. Exact solutions are given for (i) an initially uniform mixture flowing along a pipe, (ii) particles injected at a point with stream velocity into a uniformly flowing stream and (iii) for homogeneous reaction, assuming constancy of the coefficients in the transformed equation. Modifications are necessary if there is an initial distribution of particle sizes or if the coefficients are not independent of position, and possible methods, numerical and otherwise, for the treatment of practical problems of combustion and chemical reaction chambers are discussed.

A. A. Townsend (Cambridge, England).

Roberts, P. H. On the application of a statistical approximation to the theory of turbulent diffusion. J. Math. Mech. **6** (1957), 781–799.

The diffusion of a scalar property χ from a point source in stationary isotropic turbulence is considered from an Eulerian standpoint. The equations for the relevant two- and three-point correlations (and their Fourier transforms) are derived and are rendered determinate by the assumption that all fourth-order mean values are related to second-order mean values as for a normal joint probability distribution of χ and $\mathbf{u}$. The resulting system of coupled equations is very complicated and detailed results cannot easily be obtained. Some simplification is possible, however, when the viscosity and molecular diffusivity are both set equal to zero, and when the turbulence itself has reached the 'final period' of decay (i.e. when triple velocity-product mean values can be neglected); it is then possible to derive a single dynamical equation for the defining scalar of $\langle\chi u_i'\rangle_{av}$ which involves only the energy spectrum of the turbulence, and some properties of this equation are discussed. *W. H. Reid.*

Yakimov, Yu. L. An asymptotic solution involving three arbitrary functions to the equations of one-dimensional unsteady gas motion. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 937–938. (Russian)

The asymptotic solution for the spherically symmetric unsteady motion of an ideal gas is neatly written on one of the two pages of this report. The solution is found in a manner similar to the one used in calculating the asymptotic solution of a spherical blast wave [cf. Yakimov, Prikl. Mat. Meh. **19** (1955), 681–692; MR **17**, 1250].

C. D. Calsoyas (Livermore, Calif.).

Larish, E.; and Shechtman, I. On the introduction of radiation in the problems of gas dynamics. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1010–1012. (Russian)

Dans la plupart des problèmes concernant la dynamique des gaz on néglige l'influence du rayonnement.

Les auteurs exposent une méthode très simple valable pour les deux cas suivants:

1) la constante adiabatique $\kappa=\frac{4}{3}$;

2) l'on peut approcher la pression de radiation par la formule

$$p=(\kappa-1)\bar{a}T^{\kappa/\kappa-1}.$$

Les auteurs étudient comme application le cas d'une forte explosion. *M. Kiveliovitch* (Paris).

Powell, J. B. L. The diffraction of a rarefaction wave by a corner. J. Fluid Mech. **3** (1957), 243–254.

A solution is presented for the diffraction of a complete rarefaction wave, generated by the rupture of a plane membrane separating a perfect gas from a vacuum, by a wedge whose faces are inclined at small angles to the flow produced by the rarefaction wave. The solution is obtained as a first order perturbation of the complete rarefaction wave by the method of conical fields and the use of a similarity solution. The structure of the diffracted flow field is discussed, and a brief description is given of the shock waves which, in a more exact theory, would separate the various flow regimes. *J. Mahony.*

Koga, Toyoki. A method for solving problems of irrotational gas flow by means of high-speed digital computers. J. Appl. Mech. **24** (1957), 497–500.

The author proposes a method for the solution of problems of irrotational gas flow by integrating the equations from one streamline to the next. The paper describes the method for the two-dimensional steady flow, without shocks, of a perfect gas, and includes one example.

The method can be applied, regardless of the type of the fundamental equations (elliptic, parabolic, hyperbolic), but suffers from the drawback that initially, conditions such as pressure have to be given along one streamline. The author indicates the possibilities of application to other cases. *D. C. Gilles* (Glasgow).

Stanukovich, K. P. Some unsteady two- and three-dimensional gas flows. Dokl. Akad. Nauk SSSR (N.S.) 112 (1957), 595–598. (Russian)

It is shown that the construction of non-steady three-dimensional flows with one-dimensional hodographs involves selection of two arbitrary functions and three quadratures. *J. H. Giese* (Havre de Grace, Md.).

v. Krzywoblocki, M. Z.; and Hassan, H. A. Generalization of Bergman's linear integral operator method to diabatic flow. J. Soc. Indust. Appl. Math. 5 (1957), 47–65.

★ **Koga, Toyoki. Some criticism and a proposal of the fundamental equations of gas dynamics.** Proceedings of the Second Japan National Congress for Applied Mechanics, 1952, pp. 255–258. Science Council of Japan, Tokyo, 1953.

Łunc, Michał; et Luboński, Jan. Sur une solution approchée du problème de l'écoulement d'un gaz raréfié autour d'un obstacle. Arch. Mech. Stos. 8 (1956), 597–616.

Szymański, Zdzisław. Some flow problems of rarefied gases. Arch. Mech. Stos. 8 (1956), 449–470.

Jones, R. T.; and Van Dyke, M. D. The compressibility rule for drag of airfoil noses. J. Aero. Sci. 25 (1958), 171–172, 180.

The authors prove that the drag on the nose section of a parabola in subsonic compressible flow is given by $D=\frac{1}{2}\rho U^2\pi(R-s_0)(1-M^2)^{-\frac{1}{2}}$, where R is the nose radius and s_0 is the distance of the stagnation point downstream from the vertex. As a special case, $s_0=0$ when the section is at zero incidence. This formula is valid despite the presence of a stagnation point which invalidates linearized theory for detailed pressure distributions. The formula is used to check some approximate theories which use expansions in terms of M. *G. N. Lance* (Southampton).

Yang, Hsun-Tiao; and Lees, Lester. Rayleigh's problem at low Mach number according to the kinetic theory of gases. J. Math. Phys. 35 (1956), 195–235.

Čuškin, P. I. Calculation of some sonic flows of a gas. Prikl. Mat. Meh. 21 (1957), 353–360. (Russian)

Let $x+iy=\cosh(\xi+i\eta)$ and let E be the ellipse $\tanh\xi=\delta$. Consider the steady plane irrotational flow about E which, at infinity, is sonic and parallel to $y=0$. Let $\chi=H\rho u$ and $\lambda=H\rho v$, where $H^2=\sinh^2\xi+\sin^2\eta$. Let $\eta=\eta_1(\xi)$ be the first characteristic in $y>0$ that extends from E to infinity, and let $N\eta_n(\xi)=(N-n+1)\eta_1(\xi)$, $n=1, 2, \cdots, N$, for some integer N. Finally, let $\chi_n(\xi)=\chi(\xi, \eta_n)$ and $\lambda_n=\lambda(\xi, \eta_n)$. To find the flow in the region ahead of $\eta=\pm\eta_1(\xi)$, the author makes the approximations

$$\chi(\xi,\eta)=\sum_{m=0}^{N}\sum_{n=0}^{N} c_{mn}\chi_n \cos m\pi\eta/2\eta_1,$$

$$\lambda(\xi,\eta)=\sum_{m=0}^{N}\sum_{n=0}^{N} d_{mn}\lambda_n \sin m\pi\eta/2\eta_1.$$

Values of c_{mn} and d_{mn} have been tabulated for $N=1, 2, 3$. The equations of continuity and irrotationality lead to a system of ordinary differential equations for χ_n and λ_n. This has been integrated numerically on the BESM computer for $N=1, 2, 3$ for a circle and an ellipse with $\delta=2$. The results suggest $N=3$ yields sufficient accuracy. There is also a description of modifications required to adapt this procedure for flows about more general plane shapes and about ellipsoids of revolution. *J. H. Giese.*

Coburn, N. Intrinsic form of the characteristic relations in the steady supersonic flow of a compressible fluid. Quart. Appl. Math. 15 (1957), 237–248.

Dans ce travail, l'auteur donne un nouveau développement à l'étude géométrique et intrinsèque des écoulements d'un fluide parfait compressible. Il s'agit d'écoulements supersoniques, tridimensionnels, stationnaires, non nécessairement isentropiques. Les équations des caractéristiques sont écrites dans un repère local défini à partir d'une famille de surfaces caractéristiques admettant une famille de trajectoires orthogonales; les calculs sont développés à l'aide de la notation tensorielle. Les résultats généraux sont appliqués au cas où cette famille de surfaces caractéristiques est constituée par des plans parallèles. De façon plus spéciale est envisagé le cas particulier où le tourbillon est collinéaire à la vitesse. Bien que rotationnel, l'écoulement est isentropique; les lignes de courant sont des hélices le long desquelles la vitesse reste constante. *P. Germain* (Paris).

Zienkiewicz, H. K. Flow about cones at very high speeds. Aero. Quart. 8 (1957), 384–394.

When the hypersonic similarity parameter, K, the product of the tangent of the nose semi-angle and flight Mach number, exceeds about 4 for cones at zero incidence with attached shocks, the effects of vibrational excitation and dissociation become significant. However, the shock then lies close to the cone surface, and a good approximate solution is developed for the flow between the shock and cone, while the shock relations are solved numerically with the aid of tables of the thermodynamic properties of air, neglecting relaxation times. {Note that in table I, which compares this solution with that for constant specific heats, the percentage errors shown for T_w and T_s are, in fact, absolute errors.} *H. C. Levey* (Nedlands).

Helliwell, J. B. Two-dimensional flow at high subsonic speeds past wedges in channels with parallel walls. J. Fluid Mech. 3 (1958), 385–403.

The transonic approximation leading to Tricomi's equation is used to find the drag of the forebody of a wedge confined between parallel walls under two different assumptions. It is assumed either that the sonic line is normal to the wall and runs from the wall to the wedge shoulder, or that a sonic free-streamline separates from the wedge shoulder. Both models lead to similar results.

The theory is limited to fairly large values of $2K$, the ratio of wall spacing to forebody length, because if M_1 is the upstream Mach number and δ is the wedge angle, then $1-M_1$ and δ are assumed small in the theory but $K=0(\delta^{-1}(1-M_1)^{-2})$. It should be noted that A. Weinstein [Naval Ordnance Laboratory, White Oak, Md. Rep. NOLR-1132, pp. 73–82; MR **12**, 875] has already considered the first model and obtained an expression for the stream function in terms of the hodograph variables which is essentiably the same as that given here for y (proportional to ψ in this approximation). *H. C. Levey* (Nedlands).

Cheng, Sin-I. An approximate method of determining axisymmetric inviscid supersonic flow over a solid body and its wake. J. Aero. Sci. 25 (1958), 185–193.

The author presents a semi-empirical, approximate method for integrating the characteristic equation of

axisymmetric supersonic flow in regions from which the axis of symmetry is excluded. The method, which is suited for the calculation of the pressure distribution on afterbodies, has a large advantage in speed over numerical characteristics and, from the limited results quoted, would appear to be reasonably accurate. *J. J. Mahony.*

Fraenkel, L. E. The wave drag of wing-quasi-cylinder combinations at zero incidence. Aero. Quart. **9** (1958), 55–70.

By confining himself to wing-quasi-cylinder combinations, which are at zero incidence, and symmetrical about the wing plane, the author is able to derive three formulae for the wave drag of such a configuration. They are exact within the bounds of linearised theory. This is an improvement on past theories [see, e.g., Ward, Coll. Aero. Cranfield. Rep. no. **88** (1955); MR **16**, 878], which neglected the contribution to the drag arising from the interference field. Such a field must be introduced to cancel the normal velocity induced by one part of the combination at the surface of another. Of the three formulae for the drag, one involves the Laplace transform of a multisource strength function which is determined by the shape of body and wing. The other two formulae involve the multisource strength function itself. Use of the first expression avoids inversion of the influence functions determining the multisource strengths in terms of the wing and body geometry. Finally, an order of magnitude analysis is made, and this confirms the fact that Ward's theory is a good approximation for slender bodies.
G. N. Lance (Southampton).

Bender, Peter L. Diffusion of particles with memory. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 412–416.

En se basant sur les travaux de M. Kac [Proc. 2nd Berkeley Symposium Math. Statist. Probability, 1950, Univ. of California Press, 1951, pp. 189–215; MR **13**, 568], l'auteur déduit la distribution $\sigma(t)$ des temps mis par l'article avant d'être absorbé au contour Γ d'un espace euclidien à trois dimensions, ce qui permet de mettre en évidence les anomalies pouvant intervenir dans la diffusion d'un nuage particulier.

A ce titre elles présentent un intérêt certain et mériteraient d'être vérifiées en laboratoire en faisant agir différentes causes bien déterminées: photophorèse, microturbulence convective, etc. Les vérifications permettraient alors, dans certains cas, d'interpréter les temps de trajet d'un nuage particulaire en régime de diffusion moléculaire homogène, les particules jouant le rôle d'indicateurs sensibles des effets à mesurer. *M. Kiveliovitch.*

Linnell, R. D. Hypersonic flow around a sphere. J. Aero. Sci. **25** (1958), 65–66.

This note presents an estimate of the stagnation-point surface-velocity gradient and the axial shock detachment distance for hypersonic flow around a sphere. Real gas effects are taken into account. The predicted values correlate well with available experimental data which are for perfect gas conditions. *Author's summary.*

Sakurai, Akira. A note on Mott-Smith's solution of the Boltzmann equation for a shock wave. J. Fluid Mech. 3 (1957), 255–260.

Mott-Smith [Phys. Rev. (2) **82** (1951), 885–892; MR **12**, 891] simulated a strong plane shock wave as the region of interpenetration of two simple Maxwell distributions, corresponding to conditions far upstream and downstream. Their strengths were found to vary in the flow direction according to $\frac{1}{2}(1\pm\tanh 2x/X)$, where X is a function of Mach number M that measures the thickness of the shock wave. Mott-Smith gave two approximations for $X(M)$, and a third was proposed by Rosen [J. Chem. Phys. **22** (1954), 1045–1049; MR **15**, 922]. The present author shows that $X(M)$ is uniquely determined in the limit of large M. His shock thicknesses are smaller than those given before, in general accord with experiment. The possibility of finding higher terms in an asymptotic expansion for large M is considered briefly.
M. D. Van Dyke (Los Altos, Calif.).

Jukes, J. D. The structure of a shock wave in a fully ionized gas. J. Fluid Mech. 3 (1957), 275–285.

A completely ionized plasma is considered, consisting of a mixture of protons and electrons, each acting as a perfect gas in equilibrium at its own temperature. The continuum (Navier-Stokes) equations are derived for transition through a plane shock wave. Viscosity and thermal conductivity are taken as functions only of proton and of electron temperature respectively. Neglect of charge separation is justified a posteriori. The method of numerical solutions discussed is the usual one of finding integral curves joining the initial and final states [cf. Gilbarg and Paolucci, J. Rational Mech. Anal. **2** (1953), 617–642; MR **15**, 576], but now in three-space. Shock profiles calculated for Mach number 10 show that all transitions are completed in a few mean free paths, except for those of the electron temperature, which take much longer. *M. D. Van Dyke* (Los Altos, Calif.).

Whitham, G. B. A note on the stand-off distance of the shock in high speed flow past a circular cylinder. Comm. Pure Appl. Math. **10** (1957), 531–535.

Lighthill showed [J. Fluid Mech. **2** (1957), 1–32; MR **19**, 352] that at very high Mach numbers, a sphere supports a concentric spherical shock wave if the intervening flow is assumed incompressible (though irrotational). Here the corresponding result is deduced for plane flow: a circular body gives rise to a concentric circular shock. {The reviewer notes that the assumption of incompressible flow is less accurate in plane flow.} Thus, for an adiabatic exponent of 7/5 and an infinite Mach number, exact numerical solutions for a circular body give the shock-wave stand-off distance as 0.39 of the body radius, as compared with 0.26 from the present theory, whereas for the sphere, Lighthill's approximation gives a value only 9 per cent low. *M. D. Van Dyke* (Los Altos, Calif.).

Garabedian, P. R.; and Lieberstein, H. M. On the numerical calculation of detached bow shock waves in hypersonic flow. J. Aero. Sci. **25** (1958), 109–118.

Pour étudier le problème de l'onde de choc détachée, l'auteur considère celle-ci comme donnée et résout un problème de Cauchy. Le mouvement après le choc est déterminé par une équation aux dérivées partielles qui est elliptique au voisinage du sommet de l'onde; en considérant l'une des variables comme complexe et en se plaçant dans le cas où celle-ci est imaginaire pure, on est ramené à une équation hyperbolique intégrée par la méthode des différences finies; la solution est ensuite prolongée dans le plan réel de la variable en question. Le mérite de cette étude semble être de pouvoir aboutir à une définition correcte de la stabilité des donnés de Cauchy, stabilité qui doit concerner les prolongements des solutions pour les valeurs complexes des variables. *H. Cabannes.*

Guiraud, Jean-Pierre. Sur la méthode de choc-détente. C. R. Acad. Sci. Paris **245** (1957), 1778–1780.

Etant donné l'écoulement plan bidimensionnel avec onde de choc attachée d'un fluide parfait compressible, l'auteur étudie le domaine de validité de la méthode choc-détente de Eggers et Syvertson [NACA Tech. Note no. **2646** (1952); **2811** (1952); MR **13**, 882]. Si le rapport de rayons de courbure en deux points différents de l'obstacle reste voisin de l'unité, le domaine de validité est caractérisé par deux conditions; la première est verifiée si un certain paramètre est suffisamment petit sur l'onde de choc, la seconde exige que les caractéristiques (C_+) soient beaucoup plus inclinées sur l'onde de choc que les caractéristiques (C_-). *H. Cabannes* (Marseille).

Guiraud, Jean-Pierre. Ecoulements supersoniques bidimensionnels derrière une onde de choc attachée. C. R. Acad. Sci. Paris **245** (1957), 2474–2476.

La détermination de l'écoulement plan supersonique avec onde de choc attachée autour d'un profil donné est ramenée à un problème aux limites. Ce dernier consiste à trouver deux fonctions $X(\eta)$ et $Y(\xi, \eta)$ vérifiant une équation aux dérivées partielles quasi linéaire en $Y(\xi, \eta)$ et des conditions aux limites qui, pour la fonction $Y(\xi, \eta)$, définissent un problème de Cauchy; si ce problème admet une solution prolongeable jusqu'en $\eta=0$, la fonction $X(\eta)$ apparaît comme la solution d'une certaine équation fonctionnelle. Le problème aux limites ainsi posé peut être linéarisé autour d'une solution connue et le problème linéarisé peut être résolu complétement. *H. Cabannes* (Marseille).

Lal, Pyare; and Bhatnagar, P. L. Shock relations in a Fermi-Dirac gas. Proc. Nat. Inst. Sci. India. Part A. 23 (1957), 9–15.

L'auteur étudie les équations qui déterminent les discontinuités à travers une onde de choc des grandeurs qui caractérisent l'état d'un fluide partiellement dissocié; en particulier le degré de dissociation diminue à la traversée du choc. Dans les deux cas limites, gas parfait et gas totalement dissocié, on retrouve les résultats classiques. *H. Cabannes* (Marseille).

★ **Belotserkovskii, O. M. Flow past a circular cylinder with a detached shock.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 7 pp.

Translation of Dokl. Akad. Nauk SSSR **113** (1957), 509–512.

v. Krzywoblocki, M. Z. On the bounds of the thickness of a steady shock wave. Appl. Sci. Res. A. **6** (1957), 337–350.

Lamb, George L., Jr. The transmission of a spherical sound wave through a thin elastic plate. Ann. Physics **1** (1957), 233–246.

The author computes the acoustic field on one side of a thin elastic plate of infinite extent due to a point source on the other side. The problem is somewhat like the classical Sommerfeld problem in electromagnetic theory, and its solution is correspondingly similar. First, an integral representation of the sound field is deduced from the differential equations and boundary conditions; then suitable contours are found for an evaluation of the integral valid for the remote field. Although the procedure is an old one, its exposition in this paper is distinguished by clarity and freshness. *R. N. Goss.*

Olsen, Haakon; Romberg, Werner; and Wergeland, Harald. Radiation force on bodies in a sound field. J. Acoust. Soc. Amer. **30** (1958), 69–76.

The radiation force exerted by a sound field on a body fixed in space is calculated, using the equations of continuity and conservation of momentum. The momentum lost by the sound field due to the presence of the body is shown to be proportional to the time average of the force. Equations relating the force and the scattering phase shifts and scattering cross-section, respectively, are developed. The latter relation is studied for short wavelengths. All the results are considered in detail for the case of a rigid sphere.

All of the above calculations are carried out in the classical manner. A different treatment, not mentioned by the authors, of virtually the same material has been given by O. K. Mawardi [J. Phys. Radium (8) **17** (1956), 384–390; MR **17**, 1146]. *R. N. Goss.*

Lyamšev, L. M. Sound diffraction on a thin bounded elastic cylinder shell. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 271–273. (Russian)

The diffraction of a plane sound wave by a thin cylindrical shell of circular section is calculated. The shell is assumed to be contained between two rigid immovable screens normal to its axis. The radiation scattering p_s due to the shell is the integral over the surface, (*) $p_s = i\omega\rho \int_S G(r, a, \varphi-\varphi', z-z')w(\varphi', z')ds'$, where $w(\varphi, z)$ is the radial velocity component and G is the known Green's function for the wave equation $(\Delta+k^2)p=0$ in cylindrical coordinates (r, φ, z) satisfying the condition $\partial G/\partial n=0$. The velocity $w(\varphi, z)$ is determined by a system of three integro-differential equations [see E. H. Kennard, J. Appl. Mech. **20** (1953), 33–40; MR **14**, 817]. Expansion of the velocity components and the pressure in eigenfunctions of the homogeneous equations satisfying the boundary conditions at the ends $z=0$ and $z=d$ of the shell — e.g., $w(\varphi, z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} \cos m\varphi \sin(\pi nz/d)$ — allows this system to be replaced by a set of linear algebraic equations in the expansion coefficients. These are solved for a_{mn}; then $w(\varphi, z)$ is substituted in (*), and finally the integration in (*) is carried out by the saddle-point method to yield a complicated double-series representation of the desired diffraction field. *R. N. Goss.*

Raušenbah, B. V. On energy sources in the thermal excitation of sound. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 256–258. (Russian)

On the basis of elementary considerations involving only continuity and energy balance, purely thermal excitation of low-amplitude sound in a one-dimensional region of thickness $\ll\lambda/\pi$ is separated into two basic components associated with kinetic and potential energy, respectively; the corresponding phase and impedance conditions are considered. *H. G. Baerwald* (Cleveland, Ohio).

Goldberg, Z. A. On certain second-order quantities in acoustics. Akust. Ž. **3** (1957), 149–153. (Russian)

Based on the equations derived in a previous paper [Akust. Ž. **2** (1956), 325–328], the second-order term of hydrodynamic velocity, pressure, and density in a plane sound wave propagating through a viscous medium with thermal conductivity are obtained for $N\ll, \sim,$ and $\gg\nu\omega/c^2$. *H. G. Baerwald* (Cleveland, Ohio).

Lyamšev, L. M. On the theory of dispersion of sound by a thin rod. Akust. Ž. **2** (1956), 358–365. (Russian)

Meyer, Erwin; und Sessler, Gerhard. Schallausbreitung in Gasen bei hohen Frequenzen und sehr niedrigen Drucken. Z. Physik **149** (1957), 15–39.

Fay, R. D. Oppositely directed plane finite waves. J. Acoust. Soc. Amer. **29** (1957), 1200–1203.

Fay, R. D. Successful method of attack on plane progressive finite waves. J. Acoust. Soc. Amer. **28** (1956), 910–914.

Rivlin, R. S. The relation between the flow of non-Newtonian fluids and turbulent Newtonian fluids. Quart. Appl. Math. **15** (1957), 212–215.

It is suggested that the mean motion in turbulent flow of a Newtonian fluid resembles the steady flow of a non-Newtonian fluid in which the stress components may be expressed as polynomials in the gradients of velocity, acceleration, second acceleration, and so on. This assumption is equivalent to the assumption that the stress in an element of fluid at one instant of time depends on the velocity gradient at the element at that and previous instants. Results are quoted for the flow of a nearly-Newtonian fluid for which the stress matrix is given by

$$\mathbf{T}=-p\mathbf{I}+\eta\mathbf{A}_1+\varepsilon(c+\operatorname{tr}\mathbf{A}_1^2)\mathbf{A}_1^2$$

($\mathbf{A}$ is the rate of strain matrix and ε is small). For flow in a pipe of elliptical section, a secondary flow is found, similar in kind to that occurring in turbulent flow through pipes of non-circular section. Other expected phenomena are the appearance of normal stresses that are not equivalent to a hydrostatic pressure, and viscoelastic effects. It is suggested that a phenomenological theory of this kind will be entirely adequate to describe the flow properties of a turbulent fluid. *A. A. Townsend.*

Bernstein, I. B.; Frieman, E. A.; Kruskal, M. D.; and Kulsrud, R. M. An energy principle for hydromagnetic stability problems. Proc. Roy. Soc. London. Ser. A. **244** (1958), 17–40.

This investigation is concerned with the stability of an electron-positive ion plasma in a magnetic field, in which the fluid velocity at each point is assumed to vanish. It is based on a development from an energy principle first stated by Lundquist [Phys. Rev. (2) **83** (1951), 307–311; Ark. Fys. **5** (1952), 297–347; MR **13**, 189; **14**, 814], and is related to Rayleigh's principle.

The governing equations and boundary conditions are stated for a fluid satisfying certain conditions, in particular: that the fluid is perfectly conducting; that quadratic terms in velocity and current, the displacement current, and heat flow by conduction are negligible; and that the matter stress tensor is isotropic. An energy integral is obtained and a Lagrangian description given of the motion of the fluid in a small displacement from equilibrium. On this basis, a suitable energy principle is derived, which is then extended to cases where some of the above conditions are relaxed. The method is applied to find complete stability criteria for the case of plasma separated from a magnetic field by an interface, and that of a general axisymmetric system. *K. C. Westfold.*

See also: Elasticity, Plasticity: Paria; Boillet. **Optics, Electromagnetic Theory, Circuits:** Bernstein. **Classical Thermodynamics, Heat Transfer:** Tomotika and Yosinobu. **Astronomy:** Ferraro and Plumpton. **Geophysics:** Monin.

Optics, Electromagnetic Theory, Circuits

Miyamoto, Kenrô. On a comparison between wave optics and geometrical optics by using Fourier analysis. I. General theory. J. Opt. Soc. Amer. **48** (1958), 57–63.

In the first part of this paper, the geometrical optics approximation for the frequency response function of an optical system (with incoherent illumination) is derived from first principles. It is then shown that the formula is in agreement with the limiting form, as the wave length tends to zero, of a wave theoretical formula of H. H. Hopkins. It is shown further that when the wave aberrations are small in comparison with the wave length, the wave theoretical intensity distribution (I_W) in the image may be expressed as the convolution of the geometrical intensity distribution (I_G) and the intensity distribution in an aberration-free diffraction image. When the aberrations are large compared to the wave length, then $I_W \sim I_G$, if I_G is not infinite. If I_G becomes infinite, then I_W is still nearly equal to I_G, if the high frequency components are neglected. *E. Wolf* (Manchester).

Corazza, Gian Carlo; e Montebello, Carlo. Trasformate di Hankel e di Fourier nel calcolo dei diagrammi di radiazione. Boll. Un. Mat. Ital. (3) **12** (1957), 436–438.

The radiation pattern of an aperture in a plane screen is obtainable by multiplying the radiation pattern of a typical element of the aperture by a "screen factor". If the screen is in the plane $z=0$, if the aperture is a circle of unit radius, the screen factor at the point of spherical polar coordinates (r, θ, φ), where r is large, is the modulus of

$$\Psi(\theta, \varphi)=\int_0^1\int_0^{2\pi} f(\rho, \gamma)\exp[ju\rho\cos(\gamma-\varphi)]\rho\, d\rho\, d\gamma$$

where f is the real illumination function at the point of polar coordinates (ρ, γ) in the aperture and $u=2\pi\lambda^{-1}\sin\theta$ where λ is the wave-length.

If f is of the form $f_1(\rho)f_2(\gamma)$, this becomes

$$\Psi(\theta, \varphi)=\sum_{-\infty}^{\infty} j^n \exp[jn\varphi]\int_0^1 f_1(\rho)J_n(u\rho)\rho\, d\rho \int_0^{2\pi} f_2(\gamma)\exp[-jn\gamma]d\gamma,$$

which involves finite Hankel and Fourier transforms. *E. T. Copson* (St. Andrews).

Corazza, Gian Carlo; e Montebello, Carlo. Sul diagramma di radiazione di un'antenna ad apertura circolare. Boll. Un. Mat. Ital. (3) **12** (1957), 652–654.

The formula of the preceding paper is applied to the case where the illumination function $f(\rho, \gamma)$ is of the form $J_r(k_{r,p}\, \rho)\exp(jr\varphi)$, where $k_{r,p}$ is the pth root of the equation $J_r(x)=0$. *E. T. Copson* (St. Andrews).

Faure, Pierre; et Savelli, Michel. Etude théorique de la variation, en fonction de l'aire de mesure, du coefficient de Selwyn, défini à propos de la granularité des films photographiques. C. R. Acad Sci. Paris **244** (1957), 2371–2375.

If the transmittance of a photographic film at the point M is a stochastic process $T(M)$, and if $T(M)$ is to be studied experimentally by the scanning of the film by a small circle $\mathfrak{A}$ of radius a, then let $\Phi(M)$ be the mean of T over $\mathfrak{A}$. $T(M)$ and $\Phi(M)$ are assumed to be stationary and

isotropic. If it is assumed that $(\Phi-\bar{\Phi})/\bar{\Phi}$, where $\bar{\Phi}=E\{\Phi\}$, is Gaussian, then the granularity of the film is $G=[\pi a^2C(0)]^{\frac{1}{2}}/\bar{\Phi}$, where C is the autocorrelation function of $\Phi-\bar{\Phi}$ and G is independent of a. Careful experimental measurements only agree with this for sufficiently large scanning circles.

In the results announced in the present paper, the previous assumption of a Gaussian process is relaxed, and a granularity $G(a)$ obtained that is asymptotically the above G, but has a dependence on a in better agreement with experimental results. *G. L. Walker.*

Bigelmaier, Anton. Eine allgemeine Lösung des Schichtproblems der Optik. Opt. Acta **4** (1957), 81–86.

Explicit formulas are given for the transmission and reflection of a finite system of parallel thin films. Similar results have been given in several previous publications [e.g. A. W. Crook, J. Opt. Soc. Amer. **38** (1948), 954–964; MR **10**, 581]. The present paper contains some minor notational advantages. Misprints: for $-6R$ read $-6R^2$ in row 9, column δ^{14} of table 2; the term $-R\delta^6$ in the last display equation on page 85 should be $+R\delta^6$.

G. L. Walker (Southbridge, Mass.).

Bonštedt, B. É. A method of finding a wide class of electrostatic and magnetic fields for which the solutions of the basic equation of electron optics are expressed by means of known functions. Amer. Math. Soc. Transl. (2) **8** (1958), 353–356.

Translated from Ž. Tehn. Fiz. **25** (1955), 541–543 [MR **16**, 1180].

Bernstein, Ira B. Waves in a plasma in a magnetic field. Phys. Rev. (2) **109** (1958), 10–21.

This is a comprehensive study covering the following conditions: uniform and constant unperturbed electron and ion densities $N_\pm$; uniform and constant applied magnetic field B_0 (gyromagnetic frequencies $\Omega_\pm$); uniform and constant unperturbed Maxwellian velocity distributions with non-relativistic temperatures $T_\pm \ll m_\pm c^2/K$; collisions and nonlinear effects in perturbations negligible.

Perturbations are Fourier-analyzed in space (wave numbers $\mathbf{k}$) and Laplace-transformed in time. Boltzmann's equation is solved for the perturbations of the distribution function. They result from the perturbing field $\mathbf{E}$ by scalar multiplication with a vector that depends on s (the Laplace-transform variable), $\Omega_\pm$, $N_\pm$, $T_\pm$, the components of $\mathbf{k}$ and velocities $\mathbf{V}_\pm$ along and transverse to $\mathbf{B}_0$; azimuths about $\mathbf{B}_0$ are handled by an integrating-factor technique.

The resulting current density distribution $\mathbf{j}$ is substituted into the Maxwell's equation which has the form of a linear (dyadic) connexion between $\mathbf{E}$ and $\mathbf{j}$.

This leads to $\mathbf{RE}=\mathbf{a}$, where $\mathbf{a}$ summarizes initial data and $\mathbf{R}$ is a matrix with elements depending on s, $\Omega_\pm$, $N_\pm$, $T_\pm$, k and θ (angle between $\mathbf{B}_0$ and $\mathbf{k}$). The dependence on s is analytic and $\mathbf{R}$ can be continued into the left of the complex s-plane. The elements are given explicitly at the end of the paper, after introducing conjugate complex axes in the plane perpendicular to $\mathbf{B}_0$, representing right and left polarizations of $\mathbf{E}$ about $\mathbf{B}_0$.

The four-fold integration giving the components of $\mathbf{j}$ and hence the elements of $\mathbf{R}$ (the azimuthal integral and the integral in velocity space) is reduced to a single integral in an appendix, via two Bessel function identities. The detour via Bessel functions could have been avoided by changing the order of integration. A typical form of the reduced integral appears in Gordeyev's formula below.

Solving $|\mathbf{R}|=0$ for s yields the frequency spectrum, and approximations are given for different frequency ranges in relation to the various characteristic frequencies of the problem, namely ck, $\Omega_\pm$, $\nu_\pm=(KT_\pm/m_\pm)^{\frac{1}{2}}k$, $\omega_\pm=(4\pi N_\pm e^2/m_\pm)^{\frac{1}{2}}$. For $\Omega=0$, the transverse and longitudinal components (relative to $\mathbf{k}$) separate and the ionospheric dispersion formula ($\nu_\pm \ll |s|$) as well as Landau's formula for longitudinal electron-plasma oscillations result.

For $\Omega\neq 0$ and $\omega_\pm \ll |s| \ll ck$, i.e. slow (and hence longitudinal) electron-plasma oscillations, one finds the dispersion formula due to Gordeyev [Ž. Eksper. Teoret. Fiz. **23** (1952), 660–668]:

$$1+k^2a^2=s\int_0^\infty dt\exp\{-st-\tfrac{1}{2}\nu^2t^2\cos^2\theta-\nu^2\Omega^{-2}(1-\cos\Omega t)\sin^2\theta\},$$

where $a=\nu/k\omega=KT/4\pi Ne^2=$Debye length. When $\theta=\pi/2$, analytic continuation requires elimination of the poles at multiples of $i\Omega$ by a standard technique: Gross' formula [Phys. Rev. (2) **82** (1951), 232–242; MR **12**, 886] results.

By Fourier-analyzing the factor $\exp(\lambda\cos\Omega t)$ in the integrand (coefficients $I_n(\lambda)$; $\lambda=\nu^2\Omega^{-2}\sin^2\theta$), the integral is reduced to a standard form related to the error integral:

$$J(s_1'\nu')=\int_0^\infty dt\exp(-s't-\tfrac{1}{2}\nu'^2t^2)$$

(with $s'=s+\text{in}\,\Omega$, $\nu'=\nu\cos\theta$), familiar from the case $\Omega=0$. By retracing some steps used in the derivation there, it is shown that no growing solutions $\operatorname{Re} s>0$ exist. The error underlying Gordeyev's contrary conclusion is traced. W. Newcomb's general thermodynamic proof of the absence of increasing solutions is given in an appendix.

For $\theta=\pi/2$ only real frequencies are obtained (spaced at multiples of Ω, approximately), with no apparent "Landau damping", contrary to Gross' conclusions, which are refuted. If, however, one puts $\Omega=0$ without specifically choosing $\theta=\pi/2$, Landau damping appears. Bernstein attributes this discrepancy to the fact that the waves with $\theta=\pi/2$ form a set of measure zero. The reviewer sees no discrepancy since the discrete spectrum closes up to a continuum when $\theta=\pi/2$ and $\Omega\to 0$, giving an alternative representation of damped transients to Landau's.

Explicit frequencies and Landau damping are obtained for $\theta\neq\pi/2$ by using various approximations in Gordeyev's formula: Taylor development of $\cos\Omega t$ for small Ω; neglect of $\cos\Omega t$ for large Ω since its average vanishes. In the latter case, the results agree with the picture that electrons are constrained along $\mathbf{B}_0$. Again, for small enough deviations of θ from $\pi/2$ an infinite spectrum of discrete undamped oscillations results.

The inclusion of ion dynamics for slow (longitudinal) waves makes little difference except when the ions are much colder than the electrons, and B_0 is large; then one obtains slightly damped ion oscillations [cf. Spitzer, Physics of fully ionized gases, Interscience, New York, 1956], consistent with the picture that particles are constrained along $\mathbf{B}_0$. In fast waves ($|s|\sim ck$), longitudinal and transverse components become coupled via the magnetic field. The elements of $\mathbf{R}$ contain integrals similar to that in Gordyev's formula, and by making the small-Ω approximation as before, as well as using the asymptotic error integral expansion, one finds corrections to the frequencies of longitudinal plasma electron waves and transverse electromagnetic waves due to small magnetic coupling.

Finally, the general case of slow or fast ion- and electron oscillations is discussed in the limit of zero temperature. High frequency solutions (neglect of ion motion) for $\theta=0$ are as given by Spitzer. Low frequency solutions in a neutral plasma represent hydromagnetic waves (neutrality maintained). Similarly for $\theta=\pi/2$, there are high frequency solutions as obtained by Gross, as well as hydromagnetic waves. *O. Buneman.*

Boillet, Pierre. Sur l'interprétation du principe de Huygens: cas des ondes acoustiques, élastiques et électromagnétiques. Cahiers de Phys. **11** (1957), 342–368.

This third chapter of M. Boillet's paper deals with Huygens' Principle for electromagnetic waves, the discussion being very similar to that of Chapter II for elastic waves [MR **19**, 1000]. He is led to the mathematical formulation of Bromwich, which is equivalent to the physical formulation of Larmor. The magnetic charges and currents which appeared in the Bromwich-Larmor theory reappear here as analogs of the "pseudo-sources" which occurred in the formulation of Huygens' Principle for elastic waves. *E. T. Copson* (St. Andrews).

Cook, J. M. Convergence to the Møller wave-matrix. J. Math. Phys. 36 (1957), 82–87.

A wave packet comes in towards the scatterer from a great distance and recedes again after the collision. The operational formulation of the problem is the following. Let the Hamiltonian H be the self-adjoint closure of the operator $K+V$, where K is the closure of $-\nabla^2$ and V is multiplication by the real function $V(x, y, z)$, and all operators act in the Hilbert space $L_2(E_3)$. It is assumed that $V\in L_2(E_3)$. The function which represents the packet at the time t is transformed into one which represents it at the time s by the unitary operator

$$U(s, t)=e^{iKs}e^{iH(t-s)}e^{-iKt}.$$

The asymptotic behavior of the packet in the distant past or future is determined by the behavior of $U(0, t)$ for large $|t|$. In formal scattering theory it is assumed that such limits exist, though the convergence has been proved only for small perturbations V. In the present note a rigorous proof is given that $U(0, t)$ has indeed strong limits as $t\to\pm\infty$, and that these limits are isometries. Further, it is shown that $U(0, \pm\infty)K=HU(0, \pm\infty)$, and the range of $U(0, \pm\infty)$ reduces H. *E. Hille.*

Filippov, A. F. On approximate calculation of reflected and refracted waves. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 841–854. (Russian)

L'auteur établit les expressions assymptotiques au voisinage du front d'ondes pour calculer l'intensité et la forme des ondes réfléchies et réfractées. Les résultats sont valables pour n'importe quelle limite et pour une forme quelconque de l'onde incidente. La méthode utilisée diffère peu de la méthode classique de J. Hadamard. *M. Kiveliovitch* (Paris).

van Kampen, N. G. The dispersion equation for plasma waves. Physica 23 (1957), 641–650.

The various forms of the dispersion equation are discussed with reference to the simplest plasma, which is a gas consisting of classical nonrelativistic particles with a constant neutralizing background and no external fields. In circumstances where collisions are sufficiently frequent to establish local equilibrium, Maxwell's transport equations of continuity and transfer of momentum and energy are taken to a linear approximation. Then in the three-dimensional case, the dispersion formula $\omega^2=\omega_P^2+\frac{5}{3}(\kappa T/m)k^2$ is obtained; for one-dimensional variations the factor 5/3 is replaced by 3. The discrepancy with the Thomson formula, where the factor is 1, is ascribed to Thomson's assumption of isothermal conditions.

The Vlasov case, where collisions are negligible, is treated by finding stationary solutions of the perturbed equation for the velocity-distribution function, which is reduced to an eigenvalue problem. It is concluded that although, strictly speaking, no dispersion formula exists, that of Bohm and Gross (with the factor 3) is applicable in certain circumstances. *K. C. Westfold.*

Levy, Bertram R.; and Keller, Joseph B. Propagation of electromagnetic pulses around the earth. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. EM-102 (1957), i+19 pp. = Trans. I.R.E. (Antennas and propagation) **6** (1958) no. 1, 56–61.

This paper gives results which are particularly simple in the optical shadow region on or close to the surface of a sphere. The authors use a uniformly valid asymptotic expansion for the steady state, high frequency field to derive the solution for the pulse diffraction problem by Fourier superposition. The first term in a saddle point expansion is written down explicitly and numerical results are given.

There are two criticisms of details given in this paper. The authors quote certain residue series (Equations 8 and 9) which, in fact, represent only the first term of an asymptotic expansion. Similarly, after carrying out the saddle point integration, the results (Equations 21 and 22) are also given without noting that they are the leading terms of an asymptotic expansion. This point is not too important. What needs most to be examined is the fractional error of the results and the part that the large valued roots τ_s of Equation 13 ($\tau_s=O(ka)^{1/3}$) play both in the infinite series 8 and 9 and in the derived results.

V. M. Papadopoulos (Providence, R.I.).

Zharkovskii, A. G.; and Todes, O. M. Reflection of waves from an isotropic inhomogeneous layer. Soviet Physics. JETP 4 (1957), 701–703.

An approximate method is constructed for the calculation of the reflection coefficient of a plane electromagnetic wave from an isotropically inhomogeneous layer for media whose dielectric constant and conductivity depend on a single spatial coordinate. This method can be also applied to the solution of analogous acoustical problems. *Authors' summary.*

Sivkova, V. V. Computation of the vector potential of the field of a cylinder in a variable field of a solenoid. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 115–121. (Russian)

Matveev, A. N. Electron motion in synchrotrons in the presence of radiation. Nuovo Cimento (10) **6** (1957), 1305–1317.

Brehovskih, L. M. On the dispersion equation for normal waves in stratified media. Akust. Ž. **2** (1956), 341–351. (Russian)

Ufimtsev, P. J. An approximate calculation of diffraction of plane electromagnetic waves on some metallic bodies. Ž. Tehn. Fiz. 27 (1957), 1840–1849. (Russian)

As the author remarks, a rigorous solution of the diffraction problem for any but trivially simple bodies is beset with serious mathematical difficulties. In this paper he presents a method of approximating the scattering of electromagnetic waves by certain convex objects. The basis of the method is the decomposition of the scattered field into two components: a "regular" component due to the current which, at a given point in the surface of the body, would be present on a perfectly conducting plane tangent to the surface at that point; and an "irregular" component, representing the correction due to the departure of the surface from planeness.

In the vicinity of edges, the irregular component can be approximately calculated using Sommerfeld's formula for diffraction by a wedge [B. B. Baker and E. T. Copson. The mathematical theory of Huygens' principle, 2nd ed., Oxford, 1950, p. 138; for a review of the 1st ed., see MR 1, 315]. The two cases of **E**- and **H**-polarization are treated side by side. The validity of Sommerfeld's integral for various relative orientations of source, wedge, and observer is discussed. The evaluation of the integrals (by contour integration, assuming $kr \gg 1$) for all these cases yields a set of formulas constituting the solution for the irregular component, i.e., the correction which must be applied when the diffracting object has edges.

The method is applied to diffraction by an infinite plane strip of width $2a$. The field due to the regular component for $r \gg ka^2$ is found by quadratures; to this is then added the correction due to the edges, considered as half-plane wedges.

Certain important questions are left open. It is, first of all, not clear how curved surfaces and vertices can be fitted into the treatment. More important, aside from taking notice of the singularities resulting from accidents of position, the author seems to assume, without discussion, the validity of his approximation. A promised sequel, in which he is to treat diffraction by a finite cylinder, may remedy this defect. *R. N. Goss.*

Dolph, C. L. A saddle point characterization of the Schwinger stationary points in exterior scattering problems. J. Soc. Indust. Appl. Math. 5 (1957), 89–104.

This is a brief account of the Schwinger variational principles for the complex reflection coefficient in plane wave scattering from a distributed potential or an impenetrable obstacle. An important quantity is a quadratic functional defined by volume (or surface) integration of a Green's function (or its normal derivatives) and a pair of wave functions. It is shown that the imaginary part of such functionals possesses a so-called dissipative nature, namely that of being either non-negative or non-positive definite. The dissipative property allows some inferences to be made regarding the sign of the real or imaginary part of the second variation of the reflection coefficient, under special constraints in the variation of the wave function. In the absence of constraints, or other a priori knowledge, estimates of error on applying the variational principles are not generally available. *H. Levine.*

Germogenova, T. A. On solving the translation equation for strongly anisotropic scattering. Dokl. Akad. Nauk SSSR (N.S.) 113 (1957), 297–300. (Russian)

Chu, Chiao-Min. Propagation of waves in helical wave guides. J. Appl. Phys. 29 (1958), 88–99.

The author finds the field near the conductors of a helical waveguide by considering an infinite array of straight parallel wires inclined at a constant angle to a fixed axis, and supporting a periodic electromagnetic field travelling along this axis. He then adapts this solution to a helical conductor by conformal mapping. He calculates the phase velocities and also the attenuation due to finite conductivity in the wire, by the power loss method.

This solution is a new approach to an old problem; that first considered by Pocklington [Proc. Cambridge Philos. Soc. 9 (1897), 324–332]; however, the author does not seem to be aware of a good deal of work which has been done since. The reviewer has in mind the work of Brillouin [J. Appl. Phys. 19 (1948), 1023–1041], and of both Brillouin and Kornhauser in various Cruft Laboratory reports. *V. M. Papadopoulos* (Providence, R.I.).

Wait, James R. A low-frequency annular-slot antenna. J. Res. Nat. Bur. Standards 60 (1958), 59–64.

This paper deals with the low-frequency radiation characteristics of a narrow annular slot in a perfectly conducting round plane, assuming a fixed voltage between the edges. The radiation conductance is calculated for the upper half space and compared with the conductance representing power dissipation in the walls of a hemispherical cavity which backs the slot in the lower half space. A radiation efficiency is defined, and some numerical estimates given; it is suggested that the efficiency can be improved by lining the cavity wall with a wire mesh. *H. Levine* (Stanford, Calif.).

★**Nikol'skii, V. V. Gyrotropic perturbation of a waveguide.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 28 pp.

Translated from Radioteh. i Elektr., Akad. Nauk SSSR 2 (1957), 157–172.

★**Roginskii, V. N. Equivalent transformations of class II relay circuits.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 7 pp.

Translated from Dokl. Akad. Nauk SSSR 113 (1957), 328–331.

Bolinder, E. Folke. A survey of the use of non-Euclidean geometry in electrical engineering. J. Franklin Inst. 265 (1958), 169–186.

The author restricts his survey to stationary communication networks. He ignores the use of non-Euclidean geometries in stationary and rotating power-network studies which date back at least to Maxwell. *G. Kron.*

★**Lunts, A. G. Synthesis and analysis of relay-switching circuits using characteristic functions.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 6 pp.

Translated from Dokl. Akad. Nauk SSSR 75 (1950), 201–203 [MR 12, 779].

★**Tsetlin, M. L. Use of matrix calculus to synthesize relay-switching circuits.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 7 pp.

Translated from Dokl. Akad. Nauk SSSR 86 (1952), 525–528 [MR 14, 606].

★**Povarov, G. N. Matrix methods of analyzing relay-contact circuits in terms of the inoperative condition.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 6 pp.
Translated from Avtomat. i Telemeh. **15** (1954), 332–335.

★**Lunts, A. G. Application of Boolean matrix algebra to the analysis and synthesis of relay-switching circuits.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 5 pp.
Translated from Dokl. Akad. Nauk SSSR **70** (1950), 421–423 [MR **11**, 574].

★**Clavier, P. A. Some applications of the Laurent Schwartz distribution theory to network problems.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 249–265. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

Gross, B.; and Güttinger, W. Use of δ-functions in the theory of linear systems. Appl. Sci. Res. B. **6** (1956), 189–196.

Gross, Bernhard. Kanonische Darstellungen für eine Klasse verlustbehafteter Schaltungen. Arch. Elek. Übertr. **10** (1956), 299–302.

Piesch, Johanna. Über die analytische Darstellung aktiver Vierpole. Arch. Elek. Übertr. **10** (1956), 429–437.

Pipes, Louis A. Analysis of electric circuits containing nonlinear resistance. J. Franklin Inst. **263** (1957), 47–55.

Rojas Lagarde, Alfredo. On the non-reducibility of the systems of linear differential equations arising from the mesh equations of electric circuits. Rev. Un. Mat. Argentina **17** (1955), 197–204 (1956). (Spanish)

See also: Partial Order Lattices: Yablonskiĭ. **Integral Transforms:** Ragab. **Elasticity, Plasticity:** Boillet. **Relativity:** Belinfante and Swihart. **Information and Communication Theory:** Proceedings of symposium on communication theory and antenna design. **Control Systems:** Moisil; Finikov.

Classical Thermodynamics, Heat Transfer

★**Goodman, Theodore R. The heat balance integral and its application to problems involving a change of phase.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 383–400. Stanford University Press, Stanford, Calif. $7.50.

The method of the heat balance integral for solving approximately problems in heat transfer may be described roughly as follows. Such problems consist of the heat equation plus certain boundary conditions, or in cases involving a change of phase, such as melting or freezing, subsidiary differential equations. It is assumed that no heat transfer takes place outside a certain region, unknown at first and varying with time, and that inside the region the heat equation is not satisfied pointwise, but only in integrated form. The equation obtained by integrating the heat equation over the selected region is called the heat balance equation. In the one-dimensional cases considered in this paper, the temperature distribution within the region is assumed to be a quadratic, and the heat balance equation leads to an ordinary differential equation for the location of the moving boundary.

The author applies the method to a number of problems: melting of a solid with given boundary temperature or flux; melting due to aerodynamic heating; vaporization of a melting solid; etc. In cases where exact solutions are available, the results are numerically good. For example, in the simple case of temperature in a semi-infinite slab initially at constant temperature u_0 and with constant flux at $x=0$, the exact answer for the temperature at $x=0$ is $u(x, 0)=u_0+A\sqrt{t}$, where A is a constant, while the answer given by this method is $u(x, 0)=u_0+A\sqrt{(3\pi t/8)}$. The value of $\sqrt{(3\pi/8)}$ is approximately 1.09.
J. W. Green (Los Angeles, Calif.).

Tomotika, S.; and Yosinobu, H. On the convection of heat from cylinders immersed in a low-speed stream of incompressible fluid. J. Math. Phys. **36** (1957), 112–120.

A formal solution of the partial differential equation $T_{\phi\phi}+T_{\psi\psi}+2kT_\phi=0$, under the conditions $T\to 0$ as $\phi^2+\psi^2\to\infty$, $T=T_0$ when $\psi=0$, and $\phi_1<\phi<\phi_2$, is written in terms of series involving Mathieu functions. The problem arises from a transformation of the problem for steady temperatures in a stream of fluid flowing across a cylinder with steady two dimensional flow. The cylinder has an arbitrary cross section and a uniform surface temperature T_0. The authors use their solution to calculate the amount of heat lost by a unit length of the cylinder per unit time in terms of the difference $\phi_1-\phi_2$ of the velocity potential at the leading and trailing edge. Some details of transformations of variables and of boundary conditions are not discussed. *R. V. Churchill.*

Kapur, J. N. Internal ballistics of composite charges for power law of burning by the equivalent charge method. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 16–39.

The internal ballistics for the power law of burning, but for general composite charges, is here treated by the equivalent charge method, under (initially) the restrictive hypothesis that the pressure index be the same for all component charges. The results of Patni [same Proc. **21** (1955), 104–119] (for a charge of 2 components) have been somewhat improved. *A. A. Bennett.*

Kapur, J. N. Pressure-time curve in internal ballistics of solid-fuel rockets as deduced from the theory of internal ballistics of recoil-less guns. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 150–167.

Although the theory of internal ballistics of recoil-less gun and that of rockets has hitherto been developed independently of each other, this author derives the latter theory as a special case from the former. In particular he investigates the pressure-time curve till burn-out and variations in temperature thereafter. The treatment is extended to include composite charges. *A. A. Bennett.*

Kapur, J. N. Internal ballistics of a recoil-less high-low pressure gun. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 229–240.

The author develops the internal-ballistic equations for a recoil-less high-low pressure gun (of two chambers, with middle nozzle). In particular, for the isothermal

model (up to burn-out), the uniqueness of maximum pressure is established, at least when the component charges are constant-burning or degressive. *A. A. Bennett.*

Kapur, J. N. Internal ballistics of a H/L gun. Proc. Nat. Inst. Sci. India. Part A. 23 (1957), 312–321.

Explicit expressions for maximum pressure in an H/L gun are obtained, and proof established that for a cord charge, the maximum pressure must occur before all-burnt. The differential equations for the second chamber are handled in dimensionless variables. *A. A. Bennett.*

Tawakley, V. B. Effect of bore resistance on internal ballistics of guns taking into account the co-volume terms. Proc. Nat. Inst. Sci. India. Part A. 23 (1957), 274–288.

Previous theory neglected covolume terms in the equations for local bore-resistance in internal ballistics of guns using composite charges. Continuing to treat bore resistance as a point function of travel, the author takes into account covolume. Numerical integration can serve to compute over-all effects. *A. A. Bennett.*

See also: Elasticity, Plasticity: Tremmel.

Quantum Mechanics

Rodberg, Leonard S. The many-body problem and the Brueckner approximation. Ann. Physics 2 (1957), 199–225.

This review article gives a simple and clear introduction to the many body problem. It assumes no previous knowledge of the problem, and tries to avoid also the use of the formal results of scattering theory. It starts out with the formulation of the two body problem in a common external potential for which it derives the Wigner-Brillouin perturbation series. This series involves the energy of the total Hamiltonian. One of the key quantities used is the so-called effective interaction operator, whose matrix elements between eigenstates of the "free" Hamiltonian (i.e. the one not containing the interaction V_{12} between the two particles but containing the common external potential) are by definition equal to the matrix elements of V_{12} between a "free" eigenstate and an eigenstate of the total Hamiltonian. The many body problem is then attacked in a similar manner, that is, the relationship is established between the "free" configuration and the actual one. The main improvement of the present method over the Hartree approximation is its inclusion of two-body correlations. These correlations are essential when the interparticle forces are strong, and this fact makes the present method greatly superior to the Hartree approximation in cases like the nuclear field problem. The common single particle potential is determined here just as in the Hartree method by a self-consistent scheme. The scheme here, however, is somewhat more subtle because this potential also depends on the effective interaction operator. Various calculational details are also discussed, such as the removal of the implicit dependence on the total energy E (which involves a redefinition of the effective interaction operator so as to make it depend on the "free" energy rather than on E), the definition of the common potential, and the question of the unlinked clusters. This last topic was especially prone to misunderstanding and confusion in previous treatments of the subject. Finally a differential formulation of the problem is described. Two appendices give some details on the third order energy contributions and on the fourth order unlinked clusters. *M. J. Moravcsik.*

Worsley, Beatrice H. The self-consistent field with exchange for neon by FERUT program. Canad. J. Phys. 36 (1958), 289–299.

A general program is described for the computation of atomic self-consistent fields with exchange on the automatic digital computer FERUT. The procedure for obtaining improved estimates of wave functions has been made completely automatic; the energy parameters have been adjusted by a procedure involving outward integration only. The program has been written in a general form so that it may be applied to configurations containing as many as fourteen different wave functions.

Results are given for the neutral neon atom.

C. Froese (Vancouver, B.C.).

Speisman, Gerald. Convergent Schrödinger perturbation theory. Phys. Rev. (2) 107 (1957), 1180–1192.

A new formulation of Rayleigh-Schrödinger perturbation theory is given, which consists essentially of applying perturbation techniques to a selected unitary transformation of the Schrödinger equation. The particular unitary transformation is given explicitly. The formulation has several advantages over the conventional one. It provides a unified treatment of non-degenerate and degenerate theory and supplies a criterion of convergence together with bounds on the error arising from truncating the perturbation expansion at any given order. Further, it converges in many cases where the conventional expansion fails. *A. Dalgarno* (Belfast).

Kanki, T. Theory of scattering in the quantized field and Low-Chew-Wick's formalism. Nuovo Cimento (10) 6 (1957), 628–641.

In this paper the relationship between the general theory of scattering for quantized fields and the Chew-Low-Wick formalism for meson-nucleon scattering is discussed. *S. N. Gupta* (Detroit, Mich.).

Jost, R.; und Lehmann, H. Integral-Darstellung kausaler Kommutatoren. Nuovo Cimento (10) 5 (1957), 1598–1610.

Si esamina la struttura del commutatore degli operatori di campo. Si dà una rappresentazione integrale per l'elemento di matrice tra stati propri dell'operatore d'impulso, tenendo conto della causalità microscopica e dello spettro dell'operatore d'impulso stesso.

Riassunto dell'autore.

Araki, Huzihiro; Munakata, Yasuo; Kawaguchi, Masaaki; and Gotô, Tetsuo. Quantum field theory of unstable particles. Progr. Theoret. Phys. 17 (1957), 419–442.

This is one of the first papers published about the problem of an unstable particle. [Other papers published essentially simultaneously and independently about the same subject are V. Glaser and G. Källén, Nuclear Phys. 2 (1957), 706–722; T. Okabayashi and S. Sato, Progr. Theoret. Phys. 17 (1957), 30–42; MR 19, 224.] As is usually the case, the Lee model is used as an example of a theory with an unstable particle, but a more general scheme is also worked out. The main problem discussed in this paper is the question of the definition of the renormalization program for an unstable particle. The mass of such an object is not a well defined quantity, but the

exact value of it depends to a certain extent on the particular experiment chosen to define it. The arbitrariness involved here is, of course, related to the life time of the unstable particle through the uncertainty principle, i.e., the different values obtained for the mass of the unstable particle do not differ by more than the inverse life time of the particle. This fact is illustrated with the aid of very explicit and detailed computations for the case of the Lee model. A similar discussion is also made for the coupling constant renormalization and different possibilities for this procedure are reviewed.

G. Källén (Copenhagen).

Naito, Kunio. On the theory of the unstable particle in Lee's model. Progr. Theoret. Phys. **18** (1957), 200–208.

The case of an unstable V-particle in the Lee model has already been discussed by several authors [V. Glaser and G. Källén, Nuclear Phys. **2** (1957), 706–722; H. Araki, Y. Munakata, M. Kawaguchi and T. Goto, article reviewed above; T. Okabayashi and S. Sato, Progr. Theoret. Phys. **17** (1957), 30–42; MR **19**, 224]. When so much work has been done in various parts of the world about the same somewhat limited problem and published essentially simultaneously, it is perhaps unavoidable that there is much overlapping both in results and methods. The present paper investigates among other things the properties of the S-matrix. In particular, it is shown that the matrix elements $\langle V|S|N, \theta\rangle$ vanish. ($|V\rangle$ is the unstable V-particle.) This is interpreted to mean that a particle with a finite life time disintegrates during the infinite time interval that is necessary to define the S-matrix. Further investigations about this problem are promised.

G. Källén (Copenhagen).

Mohan, G. Mathematical structure of renormalizable field theories. Nuovo Cimento (10) **5** (1957), supplemento, 440–471.

This paper gives a review of some of the basic mathematical features of quantized field theories. The main new contribution of the author is a suggested method for weakening the so-called "asymptotic condition" of Lehmann, Symanzik and Zimmermann [Nuovo Cimento (10) **1** (1955), 205–225; MR **17**, 219]. The problem here is to give an exact meaning to the way in which the Heisenberg field operators approach the incoming and outgoing field operators in the remote past and in the distant future. As is well-known, this is a non-trivial problem because of the highly singular nature of the field operators and of the kernel in the integral equation relating them. The most naïve way of handling this problem is to introduce an artificial time dependence of the coupling constant in the theory (an "adiabatic switching off the interaction"), so the kernel of the integral equation becomes exponentially damped in the remote past and in the distant future. This device, which works very well in every order of perturbation theory, is, however, (for reasons that have escaped the reviewer) not considered satisfactory either by the author or by Lehmann, Symanzik and Zimmermann. Instead, they want to introduce as an explicit postulate the manner in which a certain limiting operator constructed from the Heisenberg field acts on states characterized by definite occupation numbers of the incoming particles. The specific suggestion made in this paper is the following (for particles with spin $\frac{1}{2}$). Introduce a (normalizable?) solution $u(\lambda; x)$ of the one-particle Dirac equation (λ stands for all the quantum numbers necessary to specify the solution), and consider the operators

$$a^*(\lambda) = -i\int_\tau d\sigma^\mu(x)\bar{\psi}(x)\gamma_\mu u(\lambda; x),$$

$$A^*(\lambda) = -iZ^{-\frac{1}{2}}\int_{\tau=-\infty} d\sigma^\mu(x)\bar{\Psi}(x)\gamma_\mu u(\lambda; x).$$

Here, $\psi(x)$ is the incoming field and $\Psi(x)$ is the Heisenberg field. The symbol Z is a constant to be determined later and $d\sigma^\mu(x)$ is the element of integration of a space-like surface with time parameter τ. In terms of these symbols, the asymptotic condition reads

$$a^*(\lambda)|n(\lambda)\rangle = h_n A^*(\lambda)|n(\lambda)\rangle,$$

where h_n are constants "depending only on the occupation number n of the state $|n(\lambda)\rangle$". The latter symbol is "any state including n ingoing particles of quantum numbers λ". Finally, the constant Z in the definition of $A^*(\lambda)$ is adjusted so as to make $h_0=1$. The new feature here is the possibility that the constants h_n with $n\neq 0$ may be different from one, which was not admitted in earlier work. {The reviewer here wants to remark that once it is admitted that the constants h_n could be different from one in some cases, a possibility that he very much wants to admit, he sees no fundamental reason why they could not also depend on the existence or non-existence of other particles with quantum numbers different from λ in the state $|n(\lambda)\rangle$. As far as he can see, such a weakening of the assumption would not make very much difference in the argument of the present paper, but might be important in other cases.} The author then shows in detail that he can recover most of the results that have been obtained by previous authors through the use of this stronger form of the asymptotic condition.

G. Källén (Copenhagen).

Kamefuchi, Susumu. A comment on Landau's method of integration in quantum electrodynamics. Mat.-Fys. Medd. Danske Vid. Selsk. **31** (1957), no. 6, 12 pp.

The internal consistency of quantum field theory in general and quantum electrodynamics (=Q.E.) in particular has been the subject of much discussion during the last few years. It has, e.g., been argued by Landau and collaborators [for a summary cf. L. D. Landau, Niels Bohr and the development of physics, McGraw-Hill, New York, 1955, pp. 52–69; MR **17**, 692] that one could get a more or less rigorous proof of an inconsistency of renormalized Q.E. at energies of about me^{137} (m=mass of physical electron). However, it has been pointed out that the argument of Landau involves some interchanges of various limiting procedures and that these interchanges are very hard to justify, either from the point of view of mathematical rigour, or from physical intuition. [Cf. in this connection, e.g., the review of I. Pomeranchuk, Nuovo Cimento (10) **3** (1956), 1186–1203; MR **18**, 540.] A somewhat different approach to the consistency problem of Q.E. has been proposed by the reviewer (Cf. Proceedings CERN Symposium Geneva **2** (1956), 187). This discussion starts from the intuitive idea that particles with very high energies should behave essentially as free particles. Therefore, the interaction between them should be very small and can be computed with the aid of perturbation theory. If one does this, one finds results which certainly imply serious difficulties for the theory but which are widely different from the asymptotic properties of the theory proposed by Landau. In particular, it appears that one encounters difficulties at much lower energies than indicated above. The idea that particles with very high

energies should behave as free particles is supported by every model known so far but there exists no general proof of it. Therefore, the argument based on this idea cannot claim to be very rigorous. In the paper reviewed here, Kamefuchi makes full use of the freedom in interchanging different limits that Landau allows himself. As a result he is able to show that by this technique one can actually "prove" that particles with very high energies are essentially free particles. Once this is granted, the rest of the reviewer's argument can be used. In particular, the result that one has troubles at energies much lower than me^{137} appears anew. In a certain way it thus happens that the original difficulties of Landau at the energy me^{137} do not have time to develop before they are "swamped" by other effects taking place at much lower energies. Therefore, the conclusion of Kamefuchi is that, whatever the difficulties of a quantized field theory are, they are presumably not the effects advocated by Landau.

G. Källén (Copenhagen).

Thirring, Walter E. A soluble relativistic field theory. Ann. Physics 3 (1958), 91–112.

The author solves the case of an one-dimensional relativistic Fermi field with an interaction $\lambda\bar{\psi}\psi\bar{\psi}\psi$. Writing the Lagrangian in the form $L=\frac{1}{2}(\bar{\psi}(x)\alpha^i\psi_{,i}(x)-\bar{\psi}_{,i}(x)\alpha^i\psi(x))-H^1$, where H^1 is originally unspecified, the commutation relations, the equations of motion and the canonical energy-momentum tensor are obtained. With $H^1=0$, the theory describes one kind of particle with rest mass zero. To have an interaction in the relativistic case one must have two particles at the very same place, which, however, is not possible with one kind of particle obeying Fermi statistics. Introducing now two fields, the interaction term is of the form $\lambda\psi_1{}^*\psi_2{}^*\psi_1\psi_2$. It is found that the energy commutes separately with the current integrals due to either field.

The author next constructs the eigenstates of the total energy and evaluates the matrix elements of ψ between the vacuum and a one- and a three-particle state. For point interactions, both tend to zero, but their ratio is finite. All these are obtained for the simplified case where the rest mass vanishes. This introduces an infra-red divergence in some quantities. *A. Raychaudhuri* (Calcutta).

Gupta, Suraj N. Quantum field theory in terms of ordered products. Phys. Rev. (2) 107 (1957), 1722–1726.

The usual procedure of eliminating the vacuum energy and vacuum charge in quantum field theory is to postulate the corresponding operators to be ordered products of the fields in question. The author develops a formalism by which the ordering is introduced in the Lagrangian rather than at a later stage of the development of the theory. This is accomplished by transforming the ordered products — which are defined originally only in the interaction picture [G. C. Wick, Phys. Rev. (2) 80 (1950), 268–272; MR 12, 380] — from the interaction picture to the Heisenberg picture. Obviously, the whole procedure hinges on this transformation, whose existence has never been established for the cases of interest. A consequence of the above formalism is the elimination of all "tadpole" diagrams, which, however, contribute only to the unobservable mass renormalization. *F. Rohrlich.*

Lippmann, B. A. High-energy, semiclassical scattering processes. Ann. Physics 1 (1957), 113–119.

This paper consists of two parts. The first part is formal time-independent scattering theory in terms of the resolvent $G(\lambda)=(\lambda-H)^{-1}$, where $H=H_0+V$. It is assumed that a certain separation $V=V_1+V_2$ is given and that the scattering operator S_1 for $H_1=H_0+V_1$ is known: $S_1=U_1(\lambda^*)^*V_1$, $U_1(\lambda)=1+G_1V_1$. It is shown that the scattering operator S associated with $H=H_1+V_2$ has the form $S=S_1+U_1(\lambda^*)^*V_2U$, where $U=1+GV=U_1+G_1V_2U$. This permits an iteration procedure in which each iteration contributes an additional scattering by V_2. The derivations are as usual completely formal; no attempt has been made to consider the existence of the unbounded operators involved. The second part is an application to high energy scattering. V_1 is defined to be that part of V which is responsible for forward scattering; this is accomplished by use of the WKB approximation. The final integral equation for U is seen to contain as special cases the results of Gol'dman and Migdal [Ž. Eksper. Teoret. Fiz. 28 (1955), 394–400; MR 17, 811] and of Schiff [Phys. Rev. (2) 103 (1956), 443–453; MR 17, 1261].

F. Rohrlich (Baltimore, Md.).

Seiden, Joseph. Réversibilité et irréversibilité en résonance nucléaire. C. R. Acad. Sci. Paris 243 (1956), 1308–1310.

An equation for the rate of change of the density matrix involving only diagonal matrix elements is obtained for a system of nuclear spins interacting with a lattice. The assumptions and method are given in an earlier paper by the author [J. Phys. Radium (8) 18 (1957), 173–192; MR 18, 854]. *D. L. Falkoff* (Waltham, Mass.).

Mandelstam, S. Dynamical variables in the Bethe-Salpeter formalism. Proc. Roy. Soc. London. Ser. A. 233 (1955), 248–266.

A procedure is given for evaluating the matrix elements of time ordered products of field operators between any states of the system, in terms of the propagators and Bethe-Salpeter type wave functions. The method is applied to the charge-current density of the two body bound state in the lowest approximation. The conservation of charge then leads to the orthonormalization condition on the Bethe-Salpeter wave functions. This condition may be used to determine the types of singularity which the wave functions may possess at the origin. These considerations strongly suggest that the spectrum of the Bethe-Salpeter equation is similar to that of the Dirac equation, in the sense that it changes from being discrete to continuous when the coupling constant becomes greater than a certain critical value.

The general method is extended to scattering states, and in particular, an expression is given for S-matrix elements, when some of the final particles are composite.

P. T. Matthews (London).

★Фок, В. А. [Fok, V. A.] Работы по квантовой теории поля. [Articles on quantum field theory.] Izdat. Leningrad. Univ., Leningrad, 1957. 159 pp. 11.40 rubles.

This collection contains the works of V. A. Fok on second quantization and quantum electrodynamics from 1928 to 1937.

Verde, M. The high energy limit of the potential scattering. I. Non relativistic kinematics. Nuovo Cimento (10) 6 (1957), 340–354.

The paper is concerned with asymptotic expansions at high energies in non-relativistic potential scattering. The basis of the discussion is the operator of Gel'fand and

Levitan [Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 309–360; MR **13**, 558; **17**, 489], which links the base of the Hilbert space without interaction with the base of the Hilbert space with interaction. First the phase shifts for small angular momenta and large energies are given in terms of integrals of the potential and its first derivative. Next, phase shifts for large impact parameters are given by a similar equation. The eigenfunctions for large angular momenta can then be constructed. The total scattering amplitude is also given by integrating over the impact parameter as a continuous variable. An illustrative example shows the usual diffraction patterns. Finally a formula is given which yields the potential if the phase shifts as function of impact parameter are known at a fixed large energy. *M. J. Moravcsik.*

Saxon, D. S.; and Schiff, L. I. Theory of high-energy potential scattering. Nuovo Cimento (10) **6** (1957), 614–627.

The paper is a continuation of a development by Schiff [Phys. Rev. (2) **103** (1956), 443–453; MR **17**, 1261] and Saxon [Phys. Rev. (2) **107** (1957), 871–876]. The latter formalism is now applied to the three-dimensional non-separable Schroedinger equation. An integral equation is derived by the method used by Saxon (loc. cit), and the exact scattering amplitude, differential, and total cross sections are obtained. The scattering amplitude is also expressed in terms of a more convenient integral, in which the potential is an explicit factor in the integrand. These exact results are followed by approximations which are then compared with earlier results. Estimates of errors in the approximation are also given. *M. J. Moravcsik.*

Chase, David M. One-dimensional nucleon-nuclear wall model. Phys. Rev. (2) **107** (1957), 805–819.

The bound and continuum states of a one-dimensional quantal system consisting of a particle interacting with a dynamic potential well are investigated, using various approximate methods. *A. Dalgarno* (Belfast).

Vogt, Erich; and Lascoux, Jean. Interaction of a nucleon with the nucleus. Phys. Rev. (2) **107** (1957), 1028–1040.

The paper is a more general discussion of the problem considered by Vogt [Phys. Rev. (2) **101** (1956), 1792–1798]. It is shown that the second moment of the strength function is reasonably small when one considers the nucleon-nucleon correlations and the Pauli principle. The discussion is based upon certain polarised wave functions which describe satisfactorily the correlated motion of a pair of nucleons, when they are close together, and also the motion of a single nucleon averaged over the positions of other nucleons, as in the shell model. *A. Dalgarno.*

Lippmann, B. A. Surface states in crystals. Ann. Physics **2** (1957), 16–27.

Two topics in the theory of one-dimensional crystals are discussed. It is shown that the analysis of Shockley levels and of levels originating from surface imperfections may be unified by ordering the energy bands appropriately. With this ordering, the unit cells shift as a result of band crossing, leaving an extra half-cell at each end of the crystal. These half-cells constitute impurity layers, and so give rise to surface levels.

It is shown further that if the energy is not greater than the potential everywhere in the unit cell, there are no band crossings and hence no Shockley levels; the result follows from the behaviour of Bloch functions under time and space inversion. *A. Dalgarno* (Belfast).

Chirgwin, B. H. Summation convention and the density matrix in quantum theory. Phys. Rev. (2) **107** (1957), 1013–1025.

Projection operators may be used instead of vectors to represent the states of a quantum-mechanical system. Such a formulation is developed for an N-electron system referred to a non-orthogonal set of basis functions. It is shown that in such a representation, the powerful techniques of tensor analysis, which stem from the summation convention, may be used extensively. In particular, full formulas for the reduction of the density matrix are obtained by the tensor operation of contraction of indices. It is shown that this method of reduction is equivalent to that employed by Löwdin [Phys. Rev. (2) **97** (1955), 1474–1489, 1490–1508; MR **16**, 983]. In the Hartree-Fock approximation, the one-particle reduced density matrix is idempotent. This leads to restrictions on the values of the charge and bond orders of a molecule. For butadiene, explicit expressions for these quantities are derived in terms of only two independent parameters. *A. C. Hurley.*

Liehr, Andrew D. On the use of the Born-Oppenheimer approximation in molecular problems. Ann. Physics **1** (1957), 221–232.

The Born-Oppenheimer method for solving the complete molecular Schrödinger equation is modified in such a way as to permit the use of approximate electronic wave functions. The calculation of configurational stabilities and the intensities of forbidden electronic transitions from the modified theory is discussed briefly. *A. C. Hurley.*

Koster, G. F. Matrix elements of symmetric operators. Phys. Rev. (2) **109** (1958), 227–231.

A familiar theorem relates the matrix elements of vector operators between states of definite angular momentum to the corresponding matrix elements of the angular momentum operator itself. This theorem is shown to be a special case of a group-theoretical relation which is established for any (compact) group and for operators of any irreducible symmetry. In some cases, non-trivial complications occur, the simplicity of the result for spherically symmetric systems depends upon a special property of the representations of the full rotation group. The general result involves the matrix elements in the similarity transformation which reduces a given representation of a group. It is shown how these matrix elements may be determined explicitly. *A. C. Hurley.*

Miąsek, M. The calculation of the matrix components of energy for hexagonal close-packed structure. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 805–810 (1957).

The tight binding method is applied to the hexagonal close-packed structure. The matrix components of energy between Bloch functions are expressed in terms of the general energy integrals (E-integrals). These expressions are simplified considerably by using the two centre approximation. This approximation reduces the number of different integrals from 72 to 7. *A. C. Hurley.*

Selivanenko, A. S. The exciton state of an imperfect molecular crystal. Soviet Physics. JETP **5** (1957), 79–83.

The Schrödinger equation governing the behaviour of exciton states is solved for a semi-infinite molecular

crystal. Exciton states localized at the surface are found. These states give rise to an energy band distinct from that arising from the usual volume excitons. The energy spectrum of this surface exciton band is determined, and the experimental conditions under which such a band might be observed are discussed. *A. C. Hurley.*

Artmann, Kurt. Berücksichtigung der Elektronen-Korrelation in der Quantenchemie. Z. Physik **149** (1957), 299–310.

The use of exact atomic and ionic wave functions in molecular calculations is discussed, with LiH as an example. The method advocated appears to be a special case of the method of atoms in molecules introduced by Moffitt [Proc. Roy. Soc. London. Ser. A. **210** (1951), 245–268]. *A. C. Hurley* (Melbourne).

van Kranendonk, J. Theory of induced infra-red absorption. Physica 23 (1957), 825–837.

A general formalism is developed for calculating the intensity of forbidden rotation-vibration bands induced by short range intermolecular forces. Both the total integrated absorption coefficient and the integrated absorption coefficient corresponding to definite rotational transitions are expanded in powers of the density, and the binary and ternary terms in this expansion are evaluated in terms of the distribution functions governing the positions and orientations of the molecule. It is shown that interference effects arise only for single transitions; that is, transitions involving a change of state for only one molecule. *A. C. Hurley* (Melbourne).

Companion, Audrey L.; and Ellison, Frank O. Calculation of atomic valence state energies. J. Chem. Phys. **28** (1958), 1–8.

Given a molecular wave function corresponding to a single valence bond structure, the general theory is presented for calculating either (1) the energy following complete nonadiabatic dissociation of the atoms, or (2) the intra-atomic energy of the molecule for equilibrium positions of the nuclei. In general, these energies are equal only if overlap is neglected. Tables required in this calculation are given for all cases involving s- and p-valence electrons. Promotional energies for CH and CH_2 molecules are derived for illustration. *A. C. Hurley.*

Hope, J.; and Longdon, L. W. Tensor operator methods and the tensor force. Phys. Rev. (2) **101** (1956), 710–716.

Tensor operator methods are applied to find the matrix elements of the two-nucleon tensor force between states of two inequivalent nucleons in LS coupling. The results are used to obtain the direct and exchange terms arising from a tensor-force interaction between states of a shell closed except for a single vacancy and external inequivalent nucleon. *Authors' summary.*

Froissart, Marcel; et Omnès, Roland. Sur certaines propriétés des solutions de l'équation de Chew et Low. C. R. Acad. Sci. Paris **245** (1957), 2203–2206.

On étudie les propriétés analytiques des solutions de l'équation de Chew et Low dans le cas statique de mésons pseudoscalaires symmétriques. En particulier la surface de Riemann de ces fonctions est déterminée complètement et l'on montre qu'elle comporte un nombre infini de feuillets. *Résumé de l'auteur.*

Salam, A., and Polkinghorne, J. C. On the classification of fundamental particles. Nuovo Cimento (10) **2** (1955), 685–690.

A classification of the known elementary particles is proposed in terms of the representations of the 4 dimensional Euclidean rotation group. It avoids the multiply charged states which are inherent in an earlier proposal of Pais [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 484–492; MR **16**, 320] and is very similar in physical content to the classification due to Gell-Mann [Phys. Rev. (2) **92** (1954), 833]. A form of the strong interactions which is symmetric in this four-dimensional isotopic spin space is proposed. Some emphasis is given to ascribing different quantum numbers to τ and θ-mesons, which is no longer relevant. *P. T. Matthews* (London).

Magalinskii, V. B.; and Terletskii, Ia. P. The application of the microcanonical distribution to the statistical theory of multiple production of particles. Soviet Physics. JETP **5** (1957), 483–488.

Fermi [Progr. Theoret. Phys. **5** (1950), 570–583; MR **12**, 465] studied multiple production of particles in high-energy nucleon-nucleon encounters. The a priori probabilities of formation of various numbers of particles were taken as proportional to the statistical weights of such states. Lepore and Stuart [Phys. Rev. (2) **94** (1954), 1724–1727] extended this for the extreme relativistic case to take into account conservation of energy and momentum, but not angular-momentum. Here, the microcanonical distribution is used to calculate the statistical weight of a state with an arbitrary set of particles, allowing for conservation as above, and also for Fermi-Dirac, Einstein-Bose, and Boltzmann statistics. Application is made to the creation of 1, 2 or 3 π-mesons in nucleon-nucleon collision, and further refinements and simplifications are mentioned. *C. Strachan* (Aberdeen).

Mayer, Maria Goeppert; and Telegdi, Valentine L. "Twin" neutrinos: A modified two-component theory. Phys. Rev. (2) **107** (1957), 1445–1447.

A beta-decay theory is suggested in which there are two different two-component neutrinos of opposite helicity, one occurring in S and A couplings, the other in V and T. Subsequent experiments have rendered this theory improbable. *J. C. Taylor* (London).

Sugawara, Masao. Static model in the meson theory. Progr. Theoret. Phys. **18** (1957), 383–395.

The author gives a fairly plausible treatment to show that there may exist a suitable canonical transformation which connects the relativistic pseudoscalar meson theory to the non-relativistic "static" model for pion-nucleon scattering. *S. N. Gupta* (Detroit, Mich.).

Galanin, A. D. On the possibility of formulating a meson theory with several fields. Soviet Physics. JETP **5** (1957), 460–464.

Pomeranchuk [Ž. Eksper. Teoret. Fiz. **29** (1955), 869–871] has shown that the renormalized coupling constant in pseudo-scalar meson theory tends to zero as the cut-off tends to infinity in a specified manner. Here it is shown that the same is true of more realistic meson theories, including, for instance, several mesons and fermions with different isotopic-spin properties. *J. C. Taylor.*

Gilbert, Walter. New dispersion relations for pion-nucleon scattering. Phys. Rev. (2) **108** (1957), 1078–1083.

Let $f(z)$ be an invariant scattering amplitude, where z is the pion laboratory energy divided by its mass. The new relations express the imaginary part of $f(z)/(z^2-1)^{\frac{1}{2}}$ in terms of an integral of its real part. The denominator ensures that only $\operatorname{Im} f(z)$ is required in the unphysical region. It also provides an important extra convergence factor, which makes the new relations less sensitive than the usual ones to the high energy behaviour of the amplitudes. Because of the latter property, a rather accurate value, $f^2=0.084$, of the coupling constant is obtained. The second application made is the computation of the s wave scattering lengths, a thing not possible with the ordinary dispersion relations. Because s wave contributions are small, the sum of their scattering lengths can be expressed in terms of the difference and an integral of p wave phase shifts alone. The result is in agreement with experiment. *J. C. Taylor* (London).

Mitra, A. N.; and Saxena, R. P. Meson-meson interaction in the Bethe-Salpeter approximation. Phys. Rev. (2) **108** (1957), 1083–1089.

The mutual scattering of two pions is investigated for the case in which the isotopic spin of the system has the value 0 or 2, and the mesons are in s states with respect to each other. *S. N. Gupta* (Detroit, Mich.).

Chew, G. F.; Goldberger, M. L.; Low, F. E.; and Nambu, Y. Relativistic dispersion relation approach to photomeson production. Phys. Rev. (2) **106** (1957), 1345–1355.

See also: Statistical Thermodynamics and Mechanics: Thouless. **Structure of Matter:** Deigen. **Relativity:** Belinfante and Swihart.

Relativity

Libois, P. Quelques applications des idées de Riemann sur l'espace. Schr. Forschungsinst. Math. 1 (1957), 194–201.

L'auteur analyse et commente, au point de vue de leur signification historique et de leur portée actuelle, certaines idées contenues dans le mémoire fondamental de Riemann [Abh. Ges. Wiss. Göttingen 13 (1868), 1–20]. Il developpe à cette occasion diverses conceptions personnelles relatives à la notion mathématique et physique d'espace. Signalons notamment son opinion selon laquelle un "schéma global d'univers" doit tenir compte essentiellement de deux schémas élémentaires, le schéma conforme ("univers loin de la matière", "cadre") et le schéma de Schwarzschild ("élément de base d'une construction correspondant à une répartition de matière"). *J. L. Tits* (Bruxelles).

Roy, S. R. A relativistic analogue of a simple Newtonian result. Proc. Nat. Inst. Sci. India. Part A. 23 (1957), 241–245.

Two fluids of constant densities α, β form a spherical core and a concentric spherical shell, with vacuum outside. The author uses isotropic coordinates with

$$ds^2=-e^{\mu}(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2)+e^{\nu}dt^2,$$

where μ and ν are functions of r. Without giving the details of the derivation, he states that in the shell

$$e^{\mu}=(A+Br^2)^{-2};\quad e^{\nu}=[L-M(A+Br^2)^{-1}]^2,$$

where $12AB=8\pi\beta$. In all, 9 constants appear, and the continuity of μ, ν and the pressure are used to provide a total of 8 equations connecting the constants. To get another equation, he appeals to a formula of Tolman for energy content, and thus arrives at a complete solution in terms of α, β and the radii of the two spherical surfaces. {The reviewer thinks that the above formulae for e^{μ} and e^{ν} are incorrect. If we use the Schwarzschild (r, t) coordinates, the coefficient of dr^2 is $-\lambda(1-br^2-C/r)^{-1}$, where b is proportional to the density and C is an arbitrary constant. In the core we put $C=0$ to avoid a singularity, but $C\neq 0$ in the shell, and when we transform from Schwarzschild coordinates to isotropic coordinates, the formula for e^{μ} will be much more complicated than the one given by the author.} *J. L. Synge* (Dublin).

Infeld, L. The Lagrangian with higher order derivatives and the mechanical spin of a particle. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 979–983, LXXXII. (Russian summary)

The author generalizes his previous work [same Bull. **5** (1957), 491–495; MR **19**, 814], considering now a (special) relativistic Lagrangian depending on the position x_μ, 4-velocity x_μ' and also 4-acceleration x_μ'' of a free particle. He finds the 4-momentum and angular momentum integrals and studies some simple examples of spinning particles. *F. A. E. Pirani.*

Infeld, L. The Lagrangian as a function only of coordinates and the mechanical spin of a particle. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 985–989, LXXXII. (Russian summary)

The author obtains results like those of the paper reviewed above, in a different manner, which was suggested to him by a discussion with Stueckelberg [cf. E. C. G. Stueckelberg, Helv. Phys. Acta **18** (1945), 21–44; MR **7**, 181]. He assumes here that the Lagrangian for a free particle depends only on the coordinates x_μ and on a constant vector P_μ (which corresponds to the total linear momentum, but does not satisfy $P_\mu=mx_\mu'$). The motion of spinning particles is again discussed. *F. A. E. Pirani* (Baltimore, Md.).

Chernikov, N. A. Relativistic collision integral. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 530–532. (Russian)

Boltzmann's relativistic collision integral for the motion of an α particle in an inhomogeneous medium of variable density is determined. The possibility of the decay of the particle is also taken into consideration. The result obtained is based completely on a previous paper of the same author [same Dokl. (N.S.) **112** (1957), 1030–1032]; the connection is noted only so that this paper may be more easily understood.

The problem of the collision of an α particle and a β particle under the given assumptions is here treated as a geometric problem: to determine the intersection of the life curve of β and a multidimensional strip in the state space of α. The state space of α is defined as the totality of its space-time positions and its velocities, while the life curve of β represents its trajectory between two collisions. *T. P. Anđelić* (Belgrade).

Potier, Robert. Sur l'inversion de l'axe de temps; son action sur les fonctions d'ondes et les équations d'ondes. C. R. Acad. Sci. Paris **245** (1957), 2485–2487.

La présente note étudie les représentations linéaires du groupe de Lorentz complet (comprenant les transformations impropres). Les résultats obtenus sont appliqués aux fonctions et équations d'ondes des particules.

Résumé de l'auteur.

Bażański, Stanisław. Lagrange function for the motion of charged particles in general relativity theory. II. Acta Phys. Polon. **16** (1957), 423–433. (Russian summary)

The Lagrange function is given for non-Newtonian equations of motion of charged particles. The resulting equations of motion are identical with those previously obtained by the author [same Acta **15** (1956), 363–379; MR **19**, 508]. *L. Infeld* (Warsaw).

Popovici, A. Nonlinearity of the field in conformal reciprocity theory. Soviet Physics. JETP **5** (1957), 642–651.

This is a formal paper which formulates new conformally covariant gravitational equations, describing the gravitational and electromagnetic fields respectively. From these equations the first version of nonlinear electrodynamics of Born-Infeld is deduced, as is also the variability of the gravitational constant. *L. Infeld.*

Papapetrou, A. Le problème du mouvement dans la relativité générale et dans la théorie du champ unifié d'Einstein. Ann. Inst. H. Poincaré **15** (1957), 173–203.

In this essay the problem of motion in general theory is reviewed in an easily readable form. The titles of the chapters give an idea of the contents of the paper. They are I. General conditions for the equations of motion to be deduced from the field equations. II. The astronomical problem in general relativity theory. III. The test particles in general relativity theory. IV. The problem of motion in an unified theory with non-symmetric $g_{\mu\nu}$. *L. Infeld.*

Papapetrou, A. Über periodische nichtsinguläre Lösungen in der allgemeinen Relativitätstheorie. Ann. Physik (6) **20** (1957), 399–411.

Es wird angenommen, dass in dem verwendeten de Donder'schen Koordinatensystem die Grössen $g_{\mu\nu}$ periodische Funktionen der Zeit sind, mit derselben Periode T. Die Frage die in dieser Arbeit beantwortet ist lautet: Gibt es in diesem Falle nichtsinguläre schwache Felder die die Einstein'schen Gleichungen erfüllen und im Unendlichen die Minkowski'schen Werte annehmen. Mit Hilfe des Näherungsverfahrens wird gezeigt dass es unmöglich ist alle diese Bedingungen zu erfüllen. *L. Infeld* (Warsaw).

Trautman, A. Proof of the non-existence of periodic gravitational fields representing radiation. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1115–1117, XCII. (Russian summary)

Main Theorem: The mean value of the power radiated by a periodic, asymptotically Minkowskian gravitational field is equal to zero. The field is assumed to be produced by an isolated matter distribution. The radiated power is defined to be $\int (T_0{}^k + t_0{}^k) n_k dS$, over a 2-surface at infinity in a hypersurface x^0=constant. "Asymptotically Minkowskian" means that there exist coordinate systems in which $|g_{\mu\nu}-\eta_{\mu\nu}|<M/r$ and $|\partial g_{\mu\nu}/\partial x^\rho|<M/r$ for large r (i.e., far from the distribution of matter). "Periodic" means that, in some of these coordinate systems, there is a number τ such that $g_{\mu\nu}(x^0+\tau, x^k)=g_{\mu\nu}(x^0, x^k)$. With this rigourous definition of "periodic", the result is the expected one, since it permits no secular change in the metrical properties of the source distribution. As the author points out, "the entire energy is coupled to the g-field"; thus no strictly periodic energy emission can take place.

The important question remaining is whether almost-periodic rigourous solutions of the field equations, representing radiating systems, can exist.

F. A. E. Pirani (Baltimore, Md.).

Brill, Dieter R.; and Wheeler, John A. Interaction of neutrinos and gravitational fields. Rev. Mod. Phys. **29** (1957), 465–479.

The purpose of this paper is to gain more theoretical knowledge of the behaviour of neutrinos beyond that contained in the results of elementary absorption and emission processes. Since the neutrino is not directly affected by electromagnetic fields the authors consider the neutrino in the only other available classical field, i.e. that of gravitation. This means that the neutrino equations to be investigated are certain spinor equations in Riemann space. Different equivalent formulations which are available for this purpose are briefly summarized in a mathematical section. After introducing the Dirac and the two component spinor equations, their explicit forms are obtained for a spherically symmetric metric of the usual kind, with $g_{22}=r^2$. The energy levels of an electron in an electrostatic field are compared with those in a gravitational field described by Schwarzschild's exterior solution. Approximate solutions of the neutrino equations are devised corresponding to quasi-stationary orbits of neutrinos in the interior of a thin shell spherical geon. The authors then go on to investigate the statistical mechanical equilibrium of neutrino assemblies, and the possibility of neutrino-antineutrino pair creation by interaction with the gravitational field, including the calculation of cross-sections. Thereafter they consider the stress-energy tensor of the neutrino field which must be inserted in Einstein's equations if the reaction of the neutrino field on the gravitational field is to be described. The expression obtained is used to treat the problem of the gravitational interaction of two neutrino pencils, and of the motion of a nearly bound neutrino described above. Finally, neutrino geons are briefly considered. *H. A. Buchdahl* (Hobart).

Fock, V. Three lectures on relativity theory. Rev. Mod. Phys. **29** (1957), 325–333.

This paper presents three lectures delivered by the author to the Colloquium of the Institute for Theoretical Physics, Copenhagen, Denmark, on February 18, 20, 22, 1957. The first lecture explains the motivation for the introduction of "harmonic coordinates", i.e., coordinate systems satisfying the (non-covariant) condition

$$\partial g^{\mu\nu}/\partial x^\mu=0,$$

into relativity theory. He first argues that the name General Relativity Theory for the theory usually so called is a misnomer on the grounds that the term relativity should be used in the sense of uniformity (of space-time), so that one has "relativity" when there is a transformation group such that

$$g_{\mu\nu}'(x')=g_{\mu\nu}(x).$$

Whereas in the context of galilean space-time the Lorentz

group is a group of this kind, the transformations contemplated in the general theory are not so. Once the general theory has been formulated there is no need to adhere to generally covariant equations throughout, and one may enquire whether there do not exist in a wide class of physical situations coordinate systems which are a natural generalization of the galilean system of the special theory. It is shown that harmonic coordinates are of this kind under stated conditions, and that they are uniquely defined to within a Lorentz transformation.

The second lecture presents an outline of the derivation in the first and second approximations of the equations of motion, and of their ten classical integrals, of a system of elastic bodies moving slowly enough for gravitational radiation to be negligible. The calculations are carried out under the conditions

$$\alpha \ll L \ll R \leq c/\omega,$$

where α characterises the gravitational radius of a body M, L the linear dimensions of M, R the distance of a field point from M, and ω a typical frequency associated with M, respectively. Harmonic coordinates are used throughout.

In the third lecture the condition $R \ll c/\omega$ is relaxed and (using harmonic coordinates) the author derives in outline a solution of the field equations, to the required degree of approximation, representing the emission of what are here called spherically symmetric waves from a system of moving bodies. Some remarks are added concerning the symmetrical energy-momentum pseudo-tensor and the formulation of conservation laws. *H. A. Buchdahl.*

Møller, C. On the possibility of terrestrial tests of the general theory of relativity. Nuovo Cimento (10) **6** (1957), supplemento, 381–398.

A new test of the general theory of relativity may soon be possible by the use of "atomic clocks", which now have an accuracy of 1 part in 10^{10} over long periods, and 1 in 10^{12} over shorter periods of about 1 sec. The author considers a type of atomic clock, the "maser", in which a beam of molecules maintains the oscillations. The characteristic frequency of a maser in a gravitational field is found to be in accordance with the general relativistic formula for the rate of an ideal standard clock in the field. A test of this formula, and in consequence of the equations of motion of general relativity, could be made by comparison of the rates of two masers at different heights on the earth. For a difference of height of 3 km the effect is about $(\frac{1}{3})10^{-12}$. The effect is increased by a factor of a thousand by use of an artificial satellite of highly eccentric orbit.

The possibility of measuring the effect of the earth's gravitational field on the velocity of propagation of radiation is considered, but the author states, "it does not seem possible in the foreseeable future to measure any truly relativistic effect". *C. Gilbert.*

Bergmann, Otto; and Leipnik, Roy. Space-time structure of a static spherically symmetric scalar field. Phys. Rev. (2) **107** (1957), 1157–1161.

The well-known Schwarzschild solution of Einstein's field equations depends on the special form of the energy-momentum tensor T_{ik}. The authors seek similar solutions when a more general form of T_{ik} is presupposed, namely a T_{ik} which is obtained from a particle surrounded by a spherically symmetric static field $V(r)$. While it is stated that the interpretation of the scalar field is to be left open, the suggestion is put forward that the physical property "mass" might be considered on the same level as charge, which involves the introduction of a field connected with mass particles (analogue to the structure of the field about a point electron). The field equations are obtained from the action principle

$$\delta\int (-g)^{\frac{1}{2}}(R+\mu g^{ij}V_{,i}V_{,j})d^{(4)}x=0,$$

where R is the curvature scalar, g_{ij} the metric tensor, $g=\det(g_{ij})$, while the physical meaning of μ will emerge at a later stage from a study of the equations of motion (which is reserved for a future publication). In this action principle, both V and g_{ij} are to be varied. The field equations are solved, and a 2-parameter family of line-elements and associated fields is found. The authors point out that it is doubtful whether some of these solutions are physically significant, in view of the presence of positive finite singularities. These do not appear, however, in a 2-dimensional set of solutions. The properties of these various line-elements are studied in some detail, a clear distinction being made between the so-called "point" cases and the "sphere" cases. The latter do not appear to have arisen in previous gravitational theories, and may be of interest in microscopic physics. *H. Rund.*

Regge, Tullio; and Wheeler, John A. Stability of a Schwarzschild singularity. Phys. Rev. (2) **108** (1957), 1063–1069.

The authors suggest that a spherically symmetrical object endowed with mass can be built from the mass-free Einstein's gravitational field, and it is shown that this object is stable against small nonspherical perturbations. The investigation involves considerable mathematical skill. *S. N. Gupta* (Detroit, Mich.).

Ernst, Frederick J., Jr. Linear and toroidal geons. Rev. Mod. Phys. **29** (1957), 496.

Buneman, O. Circulation: Clue to a unified theory? Nuovo Cimento (10) **5** (1957), supplemento, 92–119.

The author visualizes matter as a continuous family of unbroken world-lines. He studies the properties of particles from this view, extending the circulation formulation of electrodynamics which he had previously developed [Proc. Cambridge Philos. Soc. **50** (1954), 77–97; MR **15**, 767]. *F. A. E. Pirani* (Baltimore, Md.).

Clauser, Emilio. Movimento di particelle nel campo unitario einsteiniano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 408–416.

Du groupe (a) du système des équations du champ de la théorie d'Einstein de 1953 ((a) $R_{\underline{\alpha\beta}}=0$, $R_{(\alpha\beta,\gamma)}=0$; (b) $\Gamma_{\alpha\beta}{}^{\beta}=0$, $\mathfrak{G}^{\alpha\beta}{}_{,\gamma}=0$) l'auteur déduit une identité de Bianchi généralisée $\mathfrak{J}_{\gamma,\beta}{}^{\beta}=0$ où $\mathfrak{J}_{\gamma\beta}$ dépend des $\Gamma_{\alpha\gamma}{}^{\beta}$ et linéairement de $R_{\alpha\beta}$; $\mathfrak{J}_{\gamma}{}^{\beta}$ considéré comme pseudo-densité tensorielle énergétique est alors utilisé pour former les équations du mouvement des particules par une méthode de singularités; $\mathfrak{J}_{\gamma\beta}$ étant la divergence d'un superpotentiel $U_{\gamma}{}^{\beta\sigma}$ ces équations s'écrivent: $\int_{\sigma}(u_{\gamma}{}^{\sigma\theta}{}_{,\theta}+t_{\gamma}{}^{s})n_{s}d\sigma=0$. σ: surface renfermant une ou plusieurs singularités du champ; $t_{\gamma}{}^{\beta}$: pseudo-densité tensorielle énergétique cal- culée en tenant compte de (b). Elles ne dépendent que des singularités enfermées et sont identiquement satisfaites en l'absence de singularités. *J. Renaudie* (Rennes).

Belinfante, F. J.; and Swihart, J. C. Phenomenological linear theory of gravitation. I. Classical mechanics. Ann. Physics 1 (1957), 168–195.

Belinfante, F. J.; and Swihart, J. C. Phenomenological linear theory of gravitation. II. Interaction with the Maxwell field. Ann. Physics 1 (1957), 196-212.

Belinfante, F. J.; and Swihart, J. C. Phenomenological linear theory of gravitation. III. Interaction with the spinning electron. Ann. Physics 2 (1957), 81–99.

In the above three papers, the authors have systematically worked out a Lorentz-covariant linear theory of the gravitational field, which is able to explain the three "crucial" tests to the same extent as Einstein's theory of gravitation. The interaction of the gravitational field with the electromagnetic field and the spinning electrons is also discussed in great detail, but the gravitational field is treated throughout as a classical field.

The chief merit of these papers is to show that much of the philosophy usually associated with Einstein's theory of gravitation has very little to do with scientific reasoning. However, this work suffers from the drawback that the energy of the linear gravitational field is not positive definite. [See the article reviewed below and earlier papers by the reviewer quoted there.] *S. N. Gupta.*

Gupta, Suraj N. Einstein's and other theories of gravitation. Rev. Mod. Phys. 29 (1957), 334–336.

This paper is in the nature of a review of known work. The requirements that any theory of gravitation should (i) be Lorentz covariant, (ii) reduce to Newton's theory in the first approximation, (iii) provide a reasonable explanation of the so-called three crucial tests, are introduced. The author discusses to what extent theories whose field equations are of the type $\square U=T$ (indices suppressed), where U and T are field and source tensors respectively, can give results conforming to these requirements. Referring to his earlier papers [Proc. Phys. Soc. Sect. A. **65** (1952), 161–164, 608–619; Phys. Rev. (2) **96** (1954), 1683–1685, MR **13**, 804; **14**, 417; **16**, 532], he then goes on to show how Einstein's theory may also be regarded as a theory in flat space, if the Lagrangian be expanded as an infinite series in ascending powers of the gravitational coupling constant. Some remarks on the quantization of this theory are added. [In work of this kind one might wish for a closer discussion of the physical implications of the reinterpretation of Einstein's theory as a "theory in flat space".] *H. A. Buchdahl.*

See also: Quantum Mechanics: Thirring. **Astronomy:** Bonnor.

Astronomy

Kuzmin, G. G. Some problems concerning the dynamics of the galaxy. Izv. Akad. Nauk Ėston. SSR. Ser. Tehn. Fiz.-Mat. Nauk **1956**, 91–107. (Russian. Estonian and English summaries)

García, Godofredo. The laws and equations of Kepler for orbits with advance of perihelion without recourse to the theory of relativity. Actas Acad. Ci. Lima **20** (1957), 3–9. (Spanish)

Porfir'ev, V. V. Inner structure of a rotating star. Astr. Ž. **33** (1956), 690–697. (Russian)

Ferraro, V. C. A.; and Plumpton, C. Hydromagnetic waves in a horizontally stratified atmosphere. Astrophys. J. **127** (1958), 459–476.

"The propagation of hydromagnetic waves in a horizontally stratified atmosphere is of interest in connection with theories of sunspot magnetic fields and coronal heating. The problem was briefly discussed by Ferraro [Astrophys. J. **119** (1954), 393–406; MR **15**, 761], who assumed that both the particle velocity and the magnetic field variations were perpendicular to the magnetic field, which was taken as vertical. The resulting motion is a simple Alfvén or A-wave, in which there is equipartition of kinetic and magnetic energy. In this paper we consider the more general case, taking account of the compressibility of the gas. The associated waves, which may be termed "S-waves," are such that the particle velocity and magnetic-field variations have both horizontal and vertical components. They are of two main types, and, if the frequency is large, one of them is similar to an A-wave and the other behaves effectively as a sound wave. The A-type wave is characterized by the fact that the vertical components of the velocity and magnetic field variations are negligible. (These S-waves and A-waves are the analogous of the modified sound and Alfvén waves first discussed by van de Hulst [Symposium: Problems of cosmical aerodynamics, Central Air Documents Office, Dayton, Ohio, 1951]. This corroborates Cowling's conclusion that gravity will tend to inhibit vertical motions in a hydromagnetic wave and, ipso facto, lends support to Cowling's criticism of Alfvén's theory of sunspots. The bearing of the result on the propagation of hydromagnetic waves in the solar chromosphere and corona is discussed, and it is tentatively suggested that the solar spicules may be explained on the basis of the sonic type of S-waves". [From the author's summary.] *K. C. Westfold.*

Bonnor, W. B. Jeans' formula for gravitational instability. Monthly Not. Roy. Astr. Soc. **117** (1957), 104–117.

The author derives Jeans's formula for gravitational instability of a large mass of gas and discusses it critically. Next he generalizes the formula appropriately to the Newtonian world-models of McCrea and Milne [Quart. J. Math. Oxford Ser. **5** (1934), 73–80] by studying radial spherically symmetric perturbations of a uniform material filling an expanding universe. He finds, in agreement with earlier work by himself and others [Bonnor, Z. Astrophys. **39** (1956), 143–159; MR **17**, 1244; E. Lifshitz, Acad. Sci. USSR. J. Phys. **10** (1946), 116–129; MR **8**, 175], that condensations form too slowly to account for galaxy formation.

With additional assumptions, he extends his work to the Newtonian steady-state universe devised by McCrea [Proc. Roy. Soc. London. Ser. A. **206** (1951), 562–575; MR **12**, 866]. In this case he finds that Jeans's formula does not apply. {This result appears to the reviewer to depend rather strongly upon the particular form of the assumptions about the Newtonian steady-state model used.} In an appendix to the paper, D. W. Sciama, who had previously [Monthly Not. Roy. Astr. Soc. **115** (1955), 3–14] developed a theory of galaxy formation in the steady-state universe, points out that, in his theory, the density fluctuations leading to condensations take place not in a homogeneous universe, but in one already filled with galactic condensations, and they should not be impeded by the inapplicability of Jeans's formula. *F. A. E. Pirani* (Baltimore, Md.).

Bonnor, William Bowen. La formation des nébuleuses en cosmologie relativiste. III. Ann. Inst. H. Poincaré **15** (1957), 158–172.

Cimino, Massimo. Sulla stabilità degli ammassi globulari nella più generale ipotesi della distribuzione sferica della loro densità. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 353–357.

Scheuer, P. A. G. A statistical method for analysing observations of faint radio stars. Proc. Cambridge Philos. Soc. **53** (1957), 764–773.

See also: Mechanics of Particles and Systems: Veiga de Oliveira; Egorov. **Fluid Mechanics, Acoustics:** Lal and Bhatnagar. **Optics, Electromagnetic Theory, Circuits:** Bernstein.

Geophysics

Monin, A. S. On turbulent diffusion in the layer of the atmosphere next to the ground. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1956, 1461–1473. (Russian)

L'auteur considère l'équation de la composante de la diffusion suivant la verticale. En introduisant la stationarité on arrive à une équation du même genre que l'équation des télégraphistes. En introduisant deux paramètres sans dimension, ζ proportionnel à la hauteur et τ proportionnel à la vitesse du frottement et au temps, on ramène l'équation à une forme simple

$$\frac{\partial^2\varphi}{\partial\zeta^2}-\frac{\partial^2\varphi}{\partial\tau^2}-\frac{2z}{|\zeta|}\frac{\partial\varphi}{\partial\tau}=0 \quad (z \text{ l'altitude}).$$

L'auteur cherche à résoudre cette équation par des conditions aux limites appropriées pour différents cas particuliers.

L'auteur étudie quelques cas concrets, par exemple la propagation de la fumée dans une cheminée. L'auteur étudie également le cas de la diffusion dans une couche d'air au sol thermiquement hétérogène.

M. Kiveliovitch (Paris).

Matschinski, Matthias. Grandeurs fondamentales en météorologie et théorie des cyclones et des anticyclones. J. Sci. Météorol. 9 (1957), 61–84.

Au lieu de caractériser le champs par les cinq variables classiques: **v**, p, ρ, T et φ: vecteur vitesse, pression, densité, température et flux de chaleur qui sont données par les équations classiques de l'hydrodynamique et de la thermodynamique, l'auteur introduit les grandeurs microscopiques en essayant de construire une théorie statistique en prenant comme élément de base le tourbillon.

En introduisant un certain nombre d'hypothèses restrictives il réussit à former un système d'équations différentielles linéaires assez compliqué. L'auteur essaie d'appliquer sa théorie à la formation de cyclones et anticyclones, et aux fluctuations à grandes échelles.

M. Kiveliovitch (Paris).

Defay, Raymond. Sur la formation des germes de condensation dans la vapeur d'un corps pur. J. Sci. Météorol. 9 (1957), 39–60.

Partant de la distinction de la goutte-phase et de la goutte-particule, ou goutte embryon, l'auteur, allant au-delà de la théorie classique des fluctuations, reprend le problème de la nucléation homogène en traitant les gouttes-embryon comme un polymère de molécules associées.

A partir de là il détermine leur distribution d'équilibre et l'énergie libre de formation de la goutte-phase proprement dite, appelée elle à croître indéfiniment.

Cette théorie permet de traiter complètement la cinétique de la nucléation et l'auteur retrouve les expressions déjà données par Volmer, Becker et Döring.

M. Kiveliovitch (Paris).

Novikov, E. A. Precipitation of particles of an aerosol from the stream onto an obstacle. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 1034–1044. (Russian)

See also: Elasticity, Plasticity: Chopra.

OTHER APPLICATIONS

Economics, Management Science

Gorman, W. M. Intertemporal choice and the shape of indifference maps. Metroecon. 9 (1957), 1–22.

This paper proposes a theory of choice which satisfies the usual axioms, but also ensures the convexity of the indifference surfaces. Convexity is important both for Welfare Economics and for problems of the existence and stability of a competitive equilibrium. The proposed theory is as follows: Let x and $a_1, a_2, \cdots$ be consumption vectors and let $x=\sum\rho_i a_i$, $\rho_i\geqq 0$, $\sum\rho_i=1$. The utility function $u(x)$ is now defined as $u(x)=\max\sum\rho_i f(a_i)$, where $f(\;)$ is continuous over a closed, convex region. The function $f(\;)$ is referred to as the 'felicity function'. It is then proved that $u(x)=\max\sum\rho_i u(a_i)$, where the maximum is taken over all $a_1, a_2, \cdots, \rho_1, \rho_2, \cdots$ satisfying $x=\sum\rho_i a_i$ (bases for x).

This theory has considerable intuitive appeal. It not only leads to the convexity of indifference surfaces, but also to "diminishing marginal utility". Its implications for the demand functions are fully explored, and it is shown that by relaxing the somewhat unrealistic assumptions of independence of "felicities", strict convexity can be proved. In this case, the utility function is written $u(x)=\sum\rho_i f(a_i)+\sum C_r\delta_r^2$, where $\delta_r^2=\sum\rho_i(a_{ir}-x_r)^2$.

F. H. Hahn (Birmingham).

Cherubino, Salvatore. Matrici non negative e loro applicazioni all'economia ed alla tecnica. Statistica, Bologna **17** (1957), 349–364.

Karush, William. On a class of minimum-cost problems. Management Sci. **4** (1958), 136–153.

The production planning problem of Modigliani and Hohn [Econometrica 23 (1955). 46–66; MR **16**, 733] is to minimize $\sum_1^n[F(x_i)+A(X_i-R_i)]$, subject to the constraints $x_i\geqq 0$, $X_i\geqq R_i$, where X_i are partial sums of x_i, R_i is a given increasing sequence, and various properties (nonnegativity, convexity, continuity of derivative) are ascribed to F. The employment scheduling problem of Karush and Vazsonyi [Management Sci. 3 (1957), 140–148; Naval Res. Logist. Quart. **4** (1957), 297–320; MR **19**,

232; 19, 821] is to minimize $\sum_1^n [F(r_i, w_i)+G(w_i-w_{i-1})]$, where $G(z)=\max(\alpha z, -\beta z)$, α, $\beta\geq 0$, subject to $A(r_i)\leq w_i\leq B$, r_i given. The former balances production against inventory costs; the latter concerns smoothing an employment pattern to balance wage costs against hiring-firing costs. Both of these are special cases of the problem of minimizing $\sum_0^n P_i(z_i)+\sum_1^n Q_i(z_i-z_{i-1})$ subject to $A_i\leq z_i\leq B_i$, $a_i\leq z_i-z_{i-1}\leq b_i$, where P_i and Q_i are convex. Algorithms for the first two problems have been given in the literature cited. They are summarized here together with a procedure for solving the more general third problem. Continuous analogues are mentioned, as well as problems which combine both inventory and smoothing features. *R. Solow* (Cambridge, Mass.).

Programming, Resource Allocation, Games

Gale, David. A theorem on flows in networks. Pacific J. Math. 7 (1957), 1073–1082.

In the author's compact notation, the principal theorem of this paper is stated: Let N be a finite set, c a non-negative real- or infinite-valued function on $N\times N$, d a real function on N. A flow is an antisymmetric real function on $N\times N$ such that $f(x, y)\leq c(x, y)$ for all $(x, y)\in N\times N$. Theorem: There exists a flow f such that $\sum_{y\in N}f(y, x)\geq d(x)$ for all $x\in N$ if and only if for all $S\subseteq N$ we have $\sum_{x\in S'} d(x)\leq \sum_{x\in S', y\in S} c(y, x)$. If N consists of the nodes of a finite graph, $d(y)$ the demand at node y for an homogeneous commodity which is to flow from node x to y in amount $f(x, y)$, and $c(x, y)$ the maximum capacity for the commodity of the branch from x to y, then the theorem states that there exists a flow satisfying all the demands if and only if there is sufficient total capacity on the branches connecting any subset of nodes with its complement to carry the total demand. *P. Wolfe.*

Berge, Claude. Sur la déficience d'un réseau infini. C. R. Acad. Sci. Paris 245 (1957), 1206–1208.

The author considers a transportation network with one point of origin and a countable set Z of destinations $y_1, y_2, \cdots$ at each of which there is a positive demand $d(y_k)$. If Y is a finite subset of Z, he defines $\delta(Y)$ as the sum of the demands at the vertices of Y minus the sum of the capacities of the directed arcs entering the vertices of Y. By δ_0 is meant the least upper bound of $\delta(Y)$ for all finite subsets Y of Z. For flows with integral values, he proves the following theorems: (1) if the graph is regressively finite and if each vertex has only a finite number of entering arcs, a necessary and sufficient condition for the existence of a flow satisfying the demands is that $\delta_0=0$; (2) under the same hypotheses, δ_0 is the minimum amount by which one must diminish the demands in order to make them feasible. Theorem (1) generalizes a theorem of D. Gale for finite graphs [see the preceding review]. *M. Richardson* (Brooklyn, N.Y.).

Kuhn, H. W. Variants of the Hungarian method for assignment problems. Naval Res. Logist. Quart. 3 (1956), 253–258 (1957).

The author's "Hungarian Method" [same Quart. 2 (1955), 83–97; MR 17, 759] is a remarkably efficient algorithm based on the duality theorem of linear programming for finding that permutation $j_1, \cdots, j_n$ of integers $1, \cdots, n$ which minimizes the sum $a_{1j_1}+\cdots+a_{nj_n}$ drawn from the n by n matrix $\{a_{ij}\}$ having integral entries. Several related methods, notably that due to Ford and Fulkerson [Management Sci. 3 (1956), 24–32], have been devised for this and the more general transportation problem; another due to the author is given here. The four procedures cited in the paper are exhibited as "variants" of one another by means of a unified graphical representation of the typical computational steps. It is stated that simple examples show none of the four methods to be best for all matrices $\{a_{ij}\}$. *P. Wolfe.*

Ferguson, Allen R.; and Dantzig, George B. The allocation of aircraft to routes — an example of linear programming under uncertain demand. Management Sci. 3 (1956), 45–73.

Let K be a polyhedral convex set of points $(x_1, \cdots, x_n)$ described as the intersection of hyperplanes and half spaces. Let $k_1, \cdots, k_n$ be nonnegative constants, and let $z_1, \cdots, z_n$ be independently distributed stochastic variables with known distributions. The authors give a method, based on discrete approximations to the distributions of the z's for choosing $(x_1, \cdots, x_n)$ to maximize the expected value of $k_1\min(x_1, z_1)+\cdots+k_n\min(x_n, z_n)$.

This method is applied to a toy problem involving the allocation of aircraft of various types to various routes in order to maximize the net revenue, when the probability distributions of demand of passenger traffic for these routes are known. In this particular case, the problem is a "generalized" transportation problem, and some simplifications are possible. The authors point out that taking account of the stochastic demand does not involve very much more computational labor than is necessary in the case in which the demand is known exactly (or presumed to be exactly its mean value). *A. J. Hoffman.*

Bellman, Richard. A Markovian decision process. J. Math. Mech. 6 (1957), 679–684.

The solutions of the recurrence relation

$$f_N(i)=\operatorname*{Max}_{g}\Big[b_i(g)+\sum_{j=1}^{M} a_{ij}(g)f_{N-1}(j)\Big] \quad (i=1, \cdots, M;\ N=1, 2, \cdots)$$

are discussed; here, g is a finite-dimensional vector taking a compact (possibly finite) set of values. If $a_{ij}(g)\geq d>0$ for all i, j, and g, with $a_{i1}(g)+\cdots+a_{iM}(g)=1$ for all i and g, and if $b_i(g)$ is never negative and is positive for some i and all g, then $N^{-1}f_N(i)\to r$ when $N\to\infty$, r being a scalar quantity for which a formula is given. The problem arises in connection with the optimization of the replacement policy g in production process; the replacement process itself is Markovian. *S. K. Zaremba* (Wolverhampton).

Ward, Joe H., Jr. The counseling assignment problem. Psychometrika 23 (1958), 55–65.

The counseling assignment problem is mathematically equivalent to the Hitchcock transportation problem, which has been solved by several different methods. Given n persons to be assigned to n jobs with the productivity of the ith person on the jth job given as c_{ij}, find an assignment of persons to jobs such that the total productivity is a maximum.

The author is not satisfied with the mathematical solution of this problem in practice, on the ground that the productivity values used are usually only crude estimates, so there is "need for the intervention of counselors to account for unforeseen significant information". Accordingly, he develops a "disposition index", which can be applied in the assignment of each individual. An as-

signment which is arbitrary (as far as the mathematical condition of the problem is concerned) can then be used at any stage, and it is especially appropriate at any stage for which the disposition index indicates that any one of several alternative assignments appears to be equally useful, or almost so. The process continues with the computation of the disposition index for the remaining allocations.

The disposition index is based on a development of combinatorial theory, and provides a basis for a decision, at each stage of the assignment process, according to criteria defining the "best" assignment. Of course, there can be no guarantee that the collection of best individual assignments is the equivalent of the mathematical solution, though it sometimes is, but the process does permit the use of information available to the counselor which is not featured in the mathematical formulation of the problem. Further deviation from the mathematical solution may arise from the practice of using the same matrix of disposition indices for several assignments without computing a new index after each assignment. The disposition index is essentially a linear transform of the deviate obtained by reducing row and column means to zero. It is a transform of $c_{ij}-\bar{c}_{i\cdot}-\bar{c}_{\cdot j}+\bar{c}\cdots$ and can be identified with the contribution of a given term to the residual term in a two factor analysis of variance without replications. These terms can be used in arriving at an approximate solution by machines, as has been proposed for more general problems by other writers.

P. S. Dwyer (Ann Arbor, Mich.).

★**Karlin, Samuel. An infinite move game with a lag.** Contributions to the theory of games, vol. 3, pp. 257–272. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

This paper deals with the following game: At each unit of time, the Evader may move either one unit of distance to the left or one to the right. The Pursuer, knowing the past moves of the Evader, has one move — he predicts two units of time in advance the position of the Evader. The payoff to the Pursuer is 1 if he predicts correctly; otherwise it is zero. The author shows that this game has the value $\frac{1}{2}(3-\sqrt{5})$, and that the Evader has a unique optimal strategy which depends on the previous move only. However, the Pursuer does not have an optimal strategy.

M. Dresher (Santa Monica, Calif.).

★**Gale, David. Information in games with finite resources.** Contributions to the theory of games, vol. 3, pp. 141–145. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

In a game with finite resources, each player is required to play each pure strategy of the given matrix game a fixed number of times in any order. The payoff is the sum of the payoffs from the individual plays. The author shows that in such a game, it is of no advantage to a player to know which strategies are available to his opponent. It is also shown that the uniform mixed strategy is optimal.

M. Dresher (Santa Monica, Calif.).

★**Everett, H. Recursive games.** Contributions to the theory of games, vol. 3, pp. 47–78. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

"A recursive game is a finite set of 'game elements', which are games for which the outcome of a single play (payoff) is either a real number, or another game of the set, but not both. By assigning real numbers to game payoffs, each element of the recursive game becomes an ordinary game, whose value and optimal strategies (if they exist) of course depend upon the particular assignment. It is shown that if every game element posesses a solution for arbitrary assignments, then the recursive game possesses a solution. In particular, if the game elements possess minimax solutions for all assignments of real numbers to game payoffs, then the recursive game possesses a supinf solution in stationary strategies, while if the game elements possess only supinf solutions, then the recursive game possesses a supinf solution which may, however, require non-stationary strategies. No restrictions are placed upon the type of game elements, other than the condition that they possess solutions for arbitrary assignments of real numbers to game payoffs. Some extensions to more general games are given." [From the author's introduction.]

D. Gale.

Vogel, Walter. Die Annäherung guter Strategien bei einer gewissen Klasse von Spielen. Math. Z. 65 (1956), 283–308.

The author considers a class of continuous zero-sum two-person games on the unit square. He poses the problem of determining conditions under which an optimal strategy is the limit of a sequence of ε-optimal strategies.

A set K of functions on the unit interval is said to be uniformly T-continuous, if for every $\varepsilon>0$ there is a finite subdivision $T(\varepsilon)$ of the unit interval such that for x' and x'' in the same subinterval $|k(x')-k(x'')|<\varepsilon$ for all $k\in K$. The basic general theorem proved by Vogel is the following:

Let $K(x;y)$ be the kernel of a game on the unit square for which a value v exists. Assume $K(x;y)$ is a function of bounded variation in x for each y. Let $\{F_\varepsilon\}$ with $\varepsilon\to 0$ be a set of ε-optimal strategies for the first player, i.e.,

$$\int K(x;y)dF_\varepsilon(x)\geqq v-\varepsilon.$$

Then, if $\{F_\varepsilon\}$ is uniformly T-continuous, there is a subsequence $\{F_{\varepsilon'}\}$ with $\varepsilon'\to 0$ which converges to an optimal strategy for the first player.

This theorem is applied to a kernel of the form

$$K(x;y)=f(x)-pg(y)\begin{cases}+pf(x)g(y) & \text{for } x<y\\ +0 & \text{for } x=y\\ -f(x)g(y) & \text{for } x>y,\end{cases}$$

in which f and g are upper semi-continuous and $0\leqq f,\ g\leqq 1$, $p\geqq 0$. Games with a kernel of this form are called "duels". A monotone duel is a duel in which $f(x)$ and $g(y)$ are monotonically increasing. An α-duel is a monotone duel in which $f(x)=0$ for $x<a_1$ and $f(x)\geqq\alpha_1>0$ for $x\geqq a_1$, $g(y)=0$ for $y<a_2$ and $g(y)\geqq\alpha_2>0$ for $y\geqq a_2$.

The principal theorems concerning duels are the following. Every duel has a value. For a symmetric α-duel there exists a sequence of ε-optimal strategies which converges to an optimal strategy. For an α-duel with continuous $g(y)$ and $p\neq 0$, there exists a sequence of ε-optimal strategies for the first player which converges to an optimal strategy for the first player. In addition, it is shown how the ε-optimal strategies can be constructed for monotone duels.

An example is given of a monotone duel for which $g(y)$ is not continuous and for which optimal strategies for the first player do not exist.

E. D. Nering.

★ **Restrepo, Rodrigo. Tactical problems involving several actions.** Contributions to the theory of games, vol. 3, pp. 313–335. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

The author discusses a class of two-person, zero sum, "silent-duel" games. Each player has a finite number of chances to attain the same goal (e.g., to kill the other player), and these attempts must be made during the time interval $0 \leq t \leq 1$. The probability for success for an attempt of either player is dependent upon the time of the attempt; however, the probability functions may differ for the two players. Each player knows these functions and the total number of chances his opponent can make before the play starts. However, after the contest begins, a player does not know the number of unsuccessful attempts already made by his opponent.

The author determines a pair of optimal strategies for the game. Let P and Q be the success probabilities functions of the two players and m and n their respective possible number of chances. The solutions are characterized by two sets of numbers $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_n$. The ith action of player I must be taken in the interval $[a_i, a_{i+1}]$ (where $a_{m+1} = b_{n+1} = 1$) and the jth action of II within the interval $[b_j, b_{j+1}]$, the times of these actions being subject to probability distributions, $F_i(t)$, $G_j(t)$ $(i=1, \cdots, m, j=1, \cdots, n)$, whose densities are piecewise continuous everywhere and zero except in the intervals $[a_i, a_{i+1}]$ and $[b_j, b_{j+1}]$ respectively. It is shown that only one solution of this type exists for each $(P, Q, m, \text{and } n)$.

H. M. Gurk (Princeton, N.J.).

★ **Berkovitz, L. D.; and Fleming, W. H. On differential games with integral payoff.** Contributions to the theory of games, vol. 3, pp. 413–435. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

The authors discuss the problem of determining functions $y(t)$ and $z(t)$ which yield the max−min=min−max of the functional $J(y, z) = \int_0^T f(x, y, z)dt$, where x is determined by y and z by way of the differential equation $dx/dt = g(x, y, z)$, $x(0) = c$. Problems of this type arise in the theory of dynamic programming in connection with the study of multi-stage games of continuous type, particularly in the study of pursuit games as developed by R. Isaacs.

Here the problem is investigated by means of the analogues of the classical devices in the calculus of variations, invariant integrals, fields of extremals, etc. Under appropriate conditions, which eliminate randomization, the existence of a saddlepoint is demonstrated. A particular example is worked out in detail.

R. Bellman.

★ **Gross, O. A rational game on the square.** Contributions to the theory of games, vol. 3, pp. 307–311. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

This paper augments the study of rational games on the square initiated by Glicksberg and the author [same Contributions, vol. 2, Princeton, 1953, pp. 173–182; MR **15**, 455]. The payoff function $M(x, y) = \sum_{n=0}^{\infty} 2^{-n}\phi_n(x)\phi_n(y)$, where $\phi(x) = 2x^n - (1-x/3)^n - (x/3)^n$, is shown to be rational. The two-person zero-sum game, over the unit square having this payoff, is shown to have unique optimal mixed strategies for each player, which are both the Cantor function on [0, 1].

P. Wolfe.

★ **Walden, W. A study of simple games through experiments on computing machines.** Contributions to the theory of games, vol. 3, pp. 201–211. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

In a predecessor of this paper [P. Stein and S. Ulam, Proc. High-Speed Comput. Confer. Louisiana State Univ., Baton Rouge, La., 1955], some results were given for the experimental play of a certain type of two-person zero-sum game: Players choose alternately either of two marks; player I wins if, after n choices are made, the sequences of marks generated belongs to a preassigned subset of sequences. In the experiments related here, for each play, a winning subset was assigned by drawing at random a fraction r of the set of all such sequences. "Players" (a high-speed computer) adopted particular strategies; for example: Maximin; equidistributed random choices, chosen so that the sequences of choices already made will be an initial segment of as many desirable sequences as possible. The experiments are summarized in graphs showing the proportion of games won by one player under various combinations of strategies and values of r. *P. Wolfe* (Santa Monica, Calif.).

de Castro, Gustavo. The calculus of probability and the formalization of economic behavior. Gaz. Mat., Lisboa **15** (1955), no. 60–61, 28–32; **16** (1955), no. 62, 6–15. (Portuguese)

This is a semipopular account of the origins of the theory of probability in the study of games of chance, followed by a discussion of the economics of organized gambling and the psychology of the gambler.

T. N. E. Greville (Washington, D.C.).

Zachrisson, L. E. A tank duel with game-theoretic implications. Naval Res. Logist. Quart. **4** (1957), 131–138.

The author considers the following game: two tanks A and B approach each other along a line. When their distance apart is x, the probability that A destroys B in the time interval Δt is $p(x)\Delta t + 0(\Delta t)$; a similar function $q(x)$ is given for the probability of B destroying A. At each instant, tanks A and B are permitted to choose their speeds, u and v, respectively, where these speeds must be between given positive bounds. It is assumed that the speeds will be chosen as functions of x alone. If now A and B choose speeds $u(x)$ and $v(x)$, these, together with the functions $p(x)$ and $q(x)$, determine the probability $f(u, v)$ that A will destroy B, and f is then the payoff function of the game. The author now argues that this game is strictly determined; i.e., the function $f(u, v)$ has a saddle point. In fact, he constructs a solution to the game as follows: he defines a function $T(x)$ as the solution of a certain differential equation. Let $R(x)$ be the probability that A will eventually destroy B if they remain indefinitely at distance apart x. The optimal strategies are now for A to use his maximum speed and B his minimum speed if $R(x) < T(x)$, while A should use his minimum speed and B his maximum speed if $T(x) < R(x)$. If u and v are chosen according to this rule, then $T(x)$ turns out to be exactly the probability that A will destroy B if both have survived up to the time when their distance apart is x.

D. Gale (Santa Monica, Calif.).

Puig Adam, P. Mathematical structures in a solitaire game. Gac. Mat., Madrid (1) 9 (1957), 14–19. (Spanish)

Expository article. *J. Isbell* (Seattle, Wash.).

See also: Numerical Methods: Orchard-Hays.

Biology and Sociology

Tryon, Robert C. Communality of a variable: formulation by cluster analysis. Psychometrika 22 (1957), 241–260.

The author gives a number of equivalent definitions of the communality of a variable in factor analysis. These definitions lead to alternative methods for computing the communality from actual data, which depend on the method of factor analysis used.

C. A. B. Smith (London).

Gerard, Harold B.; and Shapiro, Harold N. Determining the degree of inconsistency in a set of paired comparisons. Psychometrika 23 (1958), 33–46.

To measure a subject's "subjective probability" X of some future event, he is presented with all possible pairs of a set of n different probability values and asked to state which of each pair better reflects his feeling toward the event. The matrix $A=(a_{ij})$, where $a_{ij}=1$ or -1, respectively, when the subject states P_i is closer to, or further from X than P_j $(a_{ii}=0)$, is called consistent if it could have arisen if X were a real number, and the subject responded in accordance with the numerical sense of nearer. Necessary and sufficient conditions for A to be consistent are given. A is inconsistent if there is at least one set of three points among which one of two relationships holds. Formulas for counting the number of triplets in each of these relationships are given. The sum of these two numbers is suggested as a measure of the degree of inconsistency. *C. H. Kraft* (East Lansing, Mich.).

Smith, C. A. B. On the estimation of intraclass correlation. Ann. Human Genetics 21 (1957), 363–373.

Rashevsky, N. A note on the geometrization of biology. Bull. Math. Biophys. 19 (1957), 201–204.

Rashevsky, N. Remark on an interesting problem in topological biology. Bull. Math. Biophys. 19 (1957), 205–208.

See also: Probability: Castoldi. **Statistics:** Aoyama. **Programming, Resource Allocation, Games:** Ward.

Information and Communication Theory

Takano, Kinsaku. On the basic theorems of information theory. Ann. Inst. Statist. Math., Tokyo 9 (1958), 53–77.

The author considers the theorems of information theory on the relation between the capacity of a noisy channel and the entropy of the source as developed by Shannon [Bell System Tech. J. 27 (1948), 379–423, 623–656; MR 10, 133], McMillan [Ann. Math. Statist. 24 (1953), 196–219; MR 14, 1101], Feinstein [Res. Lab. Electronics, Mass. Inst. Tech. Tech. Rep. No. 282 (1954); MR 17, 1098], and Hinčin [Uspehi Mat. Nauk (N.S.) 11 (1956), no. 1(67), 17–75; MR 17, 1098]. In particular, the author gives an example to show that a stationary channel with finite memory m and no foresight (anticipation) is not necessarily m-step dependent, although Hinčin implicitly proceeded as though it were. The author also reformulates Hinčin's statement of Shannon's second fundamental theorem on the existence of a code such that the rate of transmission of information is as close to H_0 as desired. To elaborate here would take up too much space.

S. Kullback (Washington, D.C.).

Féron, Robert. Information, régression, corrélation. Publ. Inst. Statist. Univ. Paris 5 (1956), 111–215.

van Soest, J. L. Signal-to-noise ratio in information. Nederl. Tijdschr. Natuurk. 22 (1956), 233–237. (Dutch)

★ **Blokh, E. L.; and Kharkevich, A. A. Geometric theory of the threshold of the capacity of a communication system.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 6 pp.

Translated from Radiotehnika 10 (1955), no. 7, 3–7.

★ **Proceedings of symposium on communication theory and antenna design.** Electronics Research Directorate, Air Force Cambridge Research Center, Air Research and Development Command, January 9, 10, 11, 1957. ii+240 pp.

The symposium was held in January 1957 in Boston, under the joint sponsorship of the Antenna Laboratory of the Air Force Cambridge Research Center and the Physical Research Laboratory of Boston University. Its main purpose was to explore the possibility of applying the concepts and techniques of communication theory to the field of antenna design. This volume contains the papers, mainly of tutorial nature, presented at the various sessions. They deal with the following subjects: Mathematical introduction (C. Bumer and F. S. Holt), Applications to electronics (A. Kohlenberg and P. Elias), Applications to optics (E. O'Neill and G. B. Parrent), Electro-optical systems in cascade (O. Schade), Applications to radio astronomy (R. N. Bracewell), and Antenna (J. Ruze, W. H. Steel and C. Drane). There are also an introduction by R. C. Gunter, summary comments by F. J. Zucker and a bibliography and attendance list.

As the titles suggest, the material covers a wide range of topics. The mathematical introduction presents a summary of the chief mathematical tool of communication theory — Fourier transforms. Applications to electronics are mainly concerned with the study of linear systems and the design of an optimum filter. The papers dealing with optics discuss imaging in terms of response functions and modern developments relating to partial coherence. The contribution concerned with radio astronomy explains methods for studying the effect of the antenna on the spatial Fourier components of the brightness temperature distribution on the source, and describes various methods of restoration. Papers which deal more specifically with antennae are concerned with the effect of aperture errors on radiation patterns and the selection of optimum antenna arrays.

Both the beginne rand the expert are likely to find

something of interest in this volume. The summary comments by F. J. Zucker should prove particularly useful; they indicate in a clear manner the similarity, as well as the differences of the various fields which, like the field of antenna design, look to communication theory for the solution of some of their problems.

E. Wolf.

Perov, V. P. The synthesis of pulse circuits and systems with a pulse feedback. Avtomat. i Telemeh. **18** (1957), 1081–1097. (Russian. English summary)

The author studies the optimum synthesis of regulatory pulse circuits which are activated at N succeeding equally spaced instants. Inputs to the circuit of the type $Y[n, \varepsilon] = U[n, \varepsilon] + p[n, \varepsilon]$ are considered, where n is the index of the last instant, and ε is the fraction of the regulatory period elapsed since the last instant. U is the "signal", and p the "disturbance". U itself consists of a stationary random component u, plus a non-random polynomial component f, of known degree r. A class of admissible circuits is delineated, and the problem is to find that circuit which, for a specified linear operator h, minimizes in a least-squares sense an "error" defined in terms of hU and the output of the circuit. Explicit computations are carried out in some of the simpler cases.

E. Reich.

★**Панов, Д. Ю. [Panov, D. Yu.] Автоматический перевод. [Machine translation.]** Izdat. Akad. Nauk SSSR, Moscow, 1956. 46 pp. 0.65 rubles.

A discussion of the problems of machine translation of English into Russian. Passages of widely different character are presented. The experiments on scientific and technical paragraphs were made on the electronic computer BESM of the Academy of Sciences of the USSR.

Luhn, H. P. A statistical approach to mechanized encoding and searching of literary information. IBM J. Res. Develop. **1** (1957), 309–317.

This is a proposal for a completely mechanized system for the recovering of information from ordinary scientific documents. The system depends largely on the discovery of a collection of "notional families" whose presence in a sentence or paragraph can be detected by the single or joint occurrence of key words which are to be listed in a large dictionary or thesaurus. (Human experts would participate in the compilation of this collection which seems related to the system of "descriptors" of Mooers [Aslib. Proc. **8**, 1, Feb. 1956].) The author describes some of the features desirable in a computer program which would encode the paragraphs of a document in terms of these "notions". Since this encoding is to be completely automatic, the author can propose the following attractive search technique: The inquirer writes an essay on the subject of interest, mentioning in his technical language everything he feels has a bearing on the subject. The machine would encode this essay, and then, on the basis of some unspecified measure of similarity, pick out those documents whose encoded descriptions match that of the inquirer's essay. It is hoped that questions of meaning can be escaped in this way; only experiment will tell. The author seems well aware that such a system will not catch documents written in an alien jargon.

M. L. Minsky.

See also: Combinatorial Analysis: Riley. **Probability:** Halphen; Pollaczek; Yaglom and Yaglom.

Control Systems

Fuller, A. T. Stability criteria for linear systems and realizability criteria for RC networks. Proc. Cambridge Philos. Soc. **53** (1957), 878–896.

Using two theorems giving necessary and sufficient conditions for the roots of two polynomial equations to be real and separated, the author derives a new set of stability criteria for linear systems. These criteria are in some cases simpler than the Hurwitz criteria. The results are applied to the realizability of RC networks.

J. Hartmanis (Schenectady, N.Y.).

Ludwig, G.; und Rollnik, H. Theorie von Regelsystemen mit zeitlich variabler Regelstärke. Z. Angew. Math. Mech. **37** (1957), 457–470. (English, French and Russian summaries)

The authors are concerned with linear control systems in which the correcting force is not a linear combination of the error $\varepsilon(t)$ and its derivatives, but rather a linear combination of $\varepsilon(t)/t$ and its derivatives. A system is desired for which the error will be reduced to zero at a prescribed instant, which is denoted as the instant $t=0$. One finds that $\varepsilon(t)$ satisfies a linear differential equation with coefficients which are polynomials in t. Most of the paper is devoted to an exposition of parts of the classical theory of such differential equations. Solutions are constructed in the form of definite integrals by Laplace's method, and various convergent power series and asymptotic series, representing the solutions, are derived. Readers must exercise caution in making use of this exposition, for there are several troublesome typographical errors. Throughout the paper, there are occasional remarks concerning the significance of the results for control theory; however, a systematic discussion of the applications of the theory is deferred to a later paper.

L. A. MacColl (New York, N.Y.).

★**Попов, Е. П. [Popov, E. P.] Динамика систем автоматического регулирования. [Dynamics of systems of automatic regulation.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 798 pp. 24.50 rubles.

Evangelisti, Giuseppe. L'analisi frequenziale nello studio dei servosistemi. Confer. Sem. Mat. Univ. Bari no. **21** (1957), 28 pp. (1 photograph).

The paper gives an introduction to the operation and the theory of linear servomechanisms. It refers especially to the application of Laplace-transformation, to the role of the frequency transfer function and to stability. The Nyquist criterion and some typical Nyquist loops are also presented. *H. Bückner* (Schenectady, N.Y.).

Vasil'ev, V. G. On evaluation of accuracy of co-reproduction of disturbances by linear servosystems and by registrating systems. Avtomat. i Telemeh. **19** (1958), 26–48. (Russian. English summary)

Necessary and sufficient conditions for accurate reproduction of the given group of disturbances by a linear reproducing system are presented. Maximum values of disturbance modulus and of its local index of enlarging are used to determine a group of disturbances.

Author's summary.

Campeau, Joseph O. The synthesis and analysis of digital systems by Boolean matrices. I. R. E. Trans. EC-6 (1957), 231–241.

A set of n Boolean functions of n variables:

$$f_i(X_1, X_2, \cdots, X_n)=Y_i;\ i=1, 2, \cdots, n,$$

representing a transformation upon the "Boolean vector" $X=(X_1, X_2, \cdots, X_n)$, is described by an $n\times 2^n$ "Boolean matrix" **B**, whose $i\times j$th element represents the jth coefficient in the canonical expansion of fi. This transformation serves to define a special form of multiplication of B by X, which, in turn, is used as a basis for defining a consistent set of matrix and vector operations.

All rules, with the exception of the rule for matrix addition, become the conventional ones, if the matrices and vectors in any formula are replaced by $2^n\times 2^n$ matrices and 2^n dimensional vectors such that the expanded columns each contain only one non-zero element {Reviewer's note}.

Problems in optimizing design and programming can be conveniently expressed in matrix notation, and illustrations are given of the application of this notation to the analysis of clocked sequential switching circuits.
D. E. Muller (Urbana, Ill.).

Moisil, Gr. C. Sur la théorie algébrique de certains circuits électriques. J. Math. Pures Appl. (9) 36 (1957), 313–324.

This is essentially a simplified version of the material contained in some of the author's papers in the series on the use of Galois' imaginaries in the theory of automata; in particular Com. Acad. R.P. Romîne 6 (1956), 505–508 [MR 19, 376]. Several examples of networks, including push-buttons and (non-polarized) relays, are worked out in detail, using the technique of Lagrange interpolation polynomials over $GF(3)$ and $GF(2)$. *E. Grosswald.*

Finikov, B. I. On a family of classes of functions in the logic algebra and their realization in the class of π-schemes. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 247–248. (Russian)

This paper provides for a certain class of Boolean functions an algorithm for the realization in parallel-series circuits (π-schemes) and upper bounds for the number of contacts used in the realizations obtained by the given algorithm. The class of functions considered is the class $R_{n,k}$ of functions of n arguments which take the value 1 for exactly k sets of those arguments. Let $L(f)$ be the minimal number of variable symbols needed to write f by means of the Boolean connectives &, V, and $^-$, and $L_k(n)$ be the maximum of $L(f)$ over the class $R_{n,k}$. It is shown that $L_k(n)\leqq 2n+k2^{k-1}$. Let $k_2, k_3, \cdots, k_n, \cdots$ be a sequence and $\varphi(n)$ be a function with $\varphi(n)\to\infty$ such that $k_n\leqq\log_2 n-\log_2\log_2 n-\varphi(n)$, then $L_{k_n}(n)$ approaches $2n$ asymptotically. Further, if N is a natural number such that $\sup_{n\to\infty}(k_n/\log_2 n)<N$, then $L_{k_n}(n)<2Nn$. Finally, if $(k_n/\log_2 n)\to\infty$, then $L_{k_n}(n)\leqq\dfrac{2k_n n}{\log_2 n}(1+o(1))$.
E. J. Cogan (Bronxville, N.Y.).

★**Ulanov, G. M. Invariance to ε in linear combination automatic regulation systems.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 6 pp.

Translated from Dokl. Akad. Nauk SSSR **112** (1957), 253–256 [MR 11, 574].

HISTORY, BIOGRAPHY

★**Делоне, Б. Н. [Delone, B. N.] Петербургская школа теории чисел [The St. Petersburg school in the theory of numbers.]** Akad. Nauk SSSR Naučno-Popularnaya Seriya. Izdat Akad. Nauk SSSR, Moscow-Leningrad, 1947. 421 pp. (6 plates) 20 rubles.

The purpose of this book is to acquaint the reader with the more important works of the six great St. Petersburg number theorists, Čebyšev, Korkin, Zolotarev, Markov, Voronoĭ and Vinogradov. A short biography is given for each, with a photograph. Two or three of the works of each are presented with some attention to original form and with a commentary. The sections on Voronoĭ and Vinogradov were written in collaboration with B. A. Venkov.

Verriest, G. La più grande scoperta dell'algebra. Civiltà delle Macchine 5 (1957), no. 4, 57–65.

A life of Galois, together with an exposition of his work and a portrait.

★**Smith, David Eugene. History of mathematics. Vol. I. General survey of the history of elementary mathematics.** Dover Publications, Inc., New York, N. Y., 1958. xxii+596 pp. $2.75.

★**Smith, David Eugene. History of mathematics. Vol. II. Special topics of elementary mathematics.** Dover Publications, Inc., New York, N. Y., 1958. xii+725 pp. $2.75.

This is an unaltered and unabridged edition of the 1951 edition (not reviewed in MR). The first edition was published in 1923. The book has been reviewed so often in other journals, and is generally so well known, as not to require review now.

Gnedenko, B. V.; and Pogrebysskу, I. B. On certain problems in the history of mathematics. Ukrain. Mat. Ž. 9 (1957), 359–368. (Russian. English summary)

Lapko, A. F.; and Lyusternik, L. A. Mathematical sessions and conferences in the USSR. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 6(78), 47–130. (Russian)

A selctive description of lectures and articles presented at the various meetings, with comment on numbers attending, historical circumstances and so forth. There are ten sections: Introduction and early period (i.e. 1917–1927); All-Russian congress in Moscow 1927; First All-Union congress in Har'kov 1930; Second All-Union congress 1934; International conferences; the Association and the pre-war Uspehi Matematičeskih Nauk; Conferences 1935–1941; war years and postwar years (1941–1949); Conferences in the fifties; Third All-Union Congress 1956.

Aleksandrov, P. S.; and Golovin, O. N. The Moscow mathematical society. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 6(78), 9–46. (Russian)

A history of the society largely by description of the work, with bibliographical detail, of its outstanding members; e.g., the first section contains a list of 114 articles presented to the society by N. E. Žukovskiĭ.

★**Struik, D. J. Het land van Stevin en Huygens. [The land of Stevin and Huygens.]** Uitgeverij Pegasus, Amsterdam, 1958. 148 pp. (16 plates)

A selective history of Dutch science (including, e.g., the work of Leeuwenhoek) but particularly of mathematics as represented by Stevin and Huygens. There are ten chapters, on the topics: the new republic, the old science, the period of transition, calculation and navigation in Holland, Simon Stevin, the new science, Descartes, Christiaan Huygens, biology, colonial science.

★**Хилькевич, Э. К. [Hil'kevič, È. K.] Геометрия Лобачевского и опыт. Философское значение творчества Лобачевского. [The geometry of Lobačevskiĭ and experience. The philosophical significance of Lobačevskiĭ's work.]** Tyumenskoe Knižnoe Izdat., Tyumen', 1956. 16 pp. (Distributed without charge.)

A popular account, chiefly of Lobacevskiĭ's discussion of parallax as related to non-Euclidean geometry, and of his effect on axiomatics and foundations of mathematics.

Aleksandrov, P. S.; Vekua, I. N.; Keldyš, M. V.; and Lavrent'ev, M. A. Vladimir Ivanovič Smirnov (on the seventieth anniversary of his birth). Uspehi Mat. Nauk (N.S.) 12 (1957), no. 6(78), 197–205. (1 plate) (Russian)

A short scientific biography, with a photograph and a bibliography of 109 entries.

Ulam, S. John von Neumann, 1903-1957. Bull. Amer. Math. Soc. 64 (1958), 1–49 (1 plate).

This is the first article in a commemorative issue in honor of von Neumann. It is a general biography, giving the setting for succeeding articles on the various aspects of his scientific work. This article contains a photograph and a bibliography with 125 entries, followed by the titles of 17 abstracts of papers presented to the American Mathematical Society.

Szénássy, Barna. Mathematical activity of Ferenc Kerekes. Acta Univ. Debrecen 3 (1956), no. 2, 3–12 (1957). (Hungarian. Russian summary)

García Rúa, J. Obituary: Teófilo Pérez-Cacho. Gac. Mat., Madrid (1) 9 (1957), 3–5 (1 plate). (Spanish)

Anonymous Obituary: Konrad Knopp. Math. Z. 67 (1957), i.

Pleskot, Václav. Obituary: Prof. Dr. Václav Hruška. Stroje na Zpracování Informací 3 (1955), 9–14 (1956). (Czech)

Dr. Václav Hruška died August 15, 1954 at the age of 66. He was Professor of applied mathematics at the Technical University, Prague, Czechoslovakia. He was deeply interested in various forms of numerical methods and published many books and papers. Many of these deal with the theory and application of nomograms and other graphical methods. *V. Vand* (University Park, Pa.).

Selberg, Arne. Obituary: Rolf Harald Gran Olsson. Norske Vid. Selsk. Forh., Trondheim 30 (1957), 71–79. (Norwegian)

A short general biography with a photograph and a bibliography of 131 entries.

MISCELLANEOUS

★**de Finetti, Bruno. Matematica logico intuitiva. Nozioni di matematiche complementari e di calcolo differenziale e integrale come introduzione agli studi di scienze economiche statistiche attuariali. 2nd ed.** Edizioni Cremonese, Rome, 1956. xxv+631 pp. 6000 Lire.

This is a slightly revised version of the first edition issued in 1944 [not reviewed in MR]. The main change consists in the addition of a set of exercises at the end of each of the eleven chapters into which the book is divided. As the author points out, the material is traditional and it provides an introduction to elementary algebra, analytical geometry and differential and integral calculus for those students of economics and statistics whose grounding in mathematics has been minimal — and forgotten.

It is natural to compare the treatment with those of G. Tintner [Mathematics and statistics for economists, Rinehart, New York, 1953] and C. A. B. Smith [Biomathematics; the principles of mathematics for students of biological science, 3rd ed., Hafner, New York, 1954]. What the American author covers in 190 pages and the Britisher in 539, the Italian requires 603 pages for. The text is thus discursive in the Continental manner but is thoroughly competent without much attempt at didactic novelty. The author introduces the idea of a function qua operation as a logical procedure early in the book and illustrates his discussions with original unusual diagrams. The chapter headings sufficiently indicate the subject matter of this introductory treatise: Logic and mathematics, The integers, Real numbers, Complex numbers, Linear systems, Analytic geometry, Limits, Derivatives, Power series, Problems in several variables, Integrals.

H. L. Seal (New York, N.Y.).

RUSSIAN TRANSLATIONS

SERIES II

Volume 1

Eleven papers on topology, function theory, and differential equations

By P. S. Aleksandrov, M. G. Kreĭn, G. E. Šilov, I. M. Gel'fand and S. V. Fomin, G. D. Suvorov, L. S. Pontryagin, I. I. Gihman, V. M. Starinskiĭ, L. Reĭzin' and I. M. Gel'fand and B. M. Levitan. Published in 1955.
304 pages **$3.60**

Volume 2

Eight papers on group theory

By A. I. Malcev, L. Ya. Kuliov, O. N. Golovin, M. A. Naĭmark, I. M. Gel'fand and M. I. Graev, and I. M. Gel'fand and Z. Ya. Šapiro. Published in 1956.
316 pages **$3.70**

Volume 3

Five papers on logic, algebra, and number theory

By A. A. Zykov, D. K. Faddeev, V. I. Nečaev, Yu. V. Linnik and A. V. Malyšev, and A. Z. Val'fiš (Walfisz). Published in 1956.
246 pages **$3.30**

Volume 4

Eight papers on algebra and number theory

By A. Z. Val'fiš (Walfisz), N. M. Korobov, and I. R. Šafarevič. Published in 1956.
237 pages **$3.80**

Volume 5

Eight papers on functional analysis and partial differential equations

By S. Ya. Havinson, M. A. Naĭmark and S. V. Fomin, M. S. Livšic, I. M. Gel'fand and D. A. Raĭkov and G. E. Šilov, I. M. Gel'fand and G. E. Šilov, A. G. Kostyučenko and G. E. Šilov, V. M. Borok and E. S. Citlanadze. Published in 1957.
333 pages **$4.30**

Volume 6

Five papers on algebra and group theory

By P. K. Raševskiĭ, E. B. Dynkin, M. A. Naĭmark, and N. Ya. Vilenkin. Published in 1957.
483 pages **$5.20**

Volume 7

Two papers on homotopy theory of continuous mappings

By M. M. Postnikov and V. G. Boltyanskiĭ. Published in 1957.
321 pages **$5.50**

25% discount to members of the American Mathematical Society.

Order from

AMERICAN MATHEMATICAL SOCIETY

190 Hope Street, Providence 6, R.I.

AUTHOR INDEX

(*Continued from cover 2*)

(*Continued on cover 4*)

AUTHOR INDEX

(Continued from cover 3)

966
943

942
1028
973
993

984
1016
1029
969
984

958
1004
998
1025
1017
945
959
1012
1026
1024

938
996
981
993
966

991
974
1007
996

949
994
977

997
014
967
942
938
987
990

005
006
985

026
966
983
939
945
948
978

011
006
999
991
996
001